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Vladimir Sosnilo

WEIGHTED METHODS IN NONCOMMUTATIVE GEOMETRY

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Academic supervisor:
Dr. Ivan Aleksandrovich Panin
corresponding member of the RAS

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Федеральное государственное бюджетное учреждение науки Санкт-Петербургское отделение Математического института им. В. А. Стеклова Российской академии наук

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Соснило Владимир Александрович

ВЕСОВЫЕ МЕТОДЫ В НЕКОММУТАТИВНОЙ ГЕОМЕТРИИ

Диссертация
на соискание ученой степени
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Панин Иван Александрович

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Contents

Introduction

Noncommutative geometry. Given a quasi-compact quasi-separated scheme X one has the ∞-category of perfect complexes $\mathbf{Perf}_{\mathrm{X}}$ associated to it. This ∞-category, in addition to being enormously useful for studying properties of X in practice, theoretically contains all the information needed to recover X:

Theorem ([Bha14, Theorem 1.5])**.** *Let* $\mathrm{Sch}_{\mathrm{qcqs}}$ *be the category of quasi-compact quasi-separated schemes and denote by* $\mathbf{Cat}_{\infty}^{\mathrm{st},\otimes}$ *the* ∞*-category of symmetric monoidal stable* ∞*-categories. The functor*

$$\begin{aligned} \mathrm{Sch}_{\mathrm{qcqs}}^{\mathrm{op}} &\to \mathbf{Cat}_{\infty}^{\mathrm{st},\otimes} \\ \mathrm{X} &\mapsto \mathbf{Perf}_{\mathrm{X}} \end{aligned}$$

is fully faithful.

On the other hand if one forgets the monoidal structure, one still has some control over functors between the underlying stable ∞-categories. For example, we have the following theorem:

Theorem ([Toë07, Corollary 1.8]; see also [Orl97, Theorem 2.2])**.** *For any smooth proper schemes* X, Y *over a ring* k *there is an equivalence*

$$\mathrm{Fun}_k(\mathbf{Perf}_{\mathrm{X}}, \mathbf{Perf}_{\mathrm{Y}}) \cong \mathbf{Perf}_{\mathrm{X}\times_k \mathrm{Y}},$$

where the left hand-side denotes the ∞*-category of exact* k*-linear functors.*

This suggests the idea of studying general stable ∞-categories instead of schemes. It is possible to extend various geometric notions and properties to this abstract context. For example, one can define reasonably well-behaved notions of regularity, smoothness and properness of k-linear stable ∞-categories for a field k [Orl16], [Lun10]. This idea is the premise for *derived noncommutative algebraic geometry* as pionereed in [KKP08]. Considering geometric problems in this more general but more flexible context found many applications back into classical algebraic geometry [Tab19], [BP21], [Per20a].

Localizing invariants. Perhaps, the most important tools in modern algebraic geometry are given by various cohomology theories, such as etale cohomology, de Rham cohomology, algebraic K-theory, motivic cohomology. Broadly speaking, these are given by taking the homotopy groups of certain nice enough functors

$$\mathrm{F}\colon\ \mathrm{Sch}_{\mathrm{qcqs}}^{\mathrm{op}} \longrightarrow \mathrm{Spt}.$$

A miraculously useful observation is that in many examples, including algebraic K-theory, topological cyclic and Hochschild homology, topological K-theory and (the connective cover of) etale K-theory (see [BGT13], [Bla16] and [CM19]), these functors extend to functors

$$\mathbf{Cat}_{\infty}^{\mathrm{perf}} \longrightarrow \mathrm{Spt}$$

in a way making the diagram

$$\begin{array}{ccc} \mathrm{Sch}_{\mathrm{qcqs}}^{\mathrm{op}} & \longrightarrow & \mathrm{Spt} \\ {\scriptstyle \mathbf{Perf}}\downarrow & \nearrow{\scriptstyle \mathrm{F}} & \\ \mathbf{Cat}_{\infty}^{\mathrm{perf}} & & \end{array}$$

commutative. Moreover, it is often the case that F is a *localizing invariant*, i.e. it sends Karoubi-Verdier sequences[1] of stable ∞-categories into fiber sequences of spectra.

[1] i.e. diagrams $\mathcal{A} \xrightarrow{i} \mathcal{B} \xrightarrow{p} \mathcal{C}$ such that $p \circ i \simeq 0$ and the induced functor $\mathcal{B}/\mathcal{A} \to \mathcal{C}$ is an equivalence after idempotent completion

The extra functoriality of localizing invariants sometimes allows one to obtain properties and computations of cohomology theories, even when it is very hard to do via more context-specific geometric methods. We provide a couple of examples for this claim:

- In [LT19] Land and Tamme proved the so-called *pro-cdh descent* property for all localizing invariants. We note that the particular case $\mathrm{F} = \mathrm{K}$ of this result was one of the crucial ingredients in the course of proving the Weibel's conjecture about vanishing of the negative K-theory [KST17].
- Kaledin [Kal17], [Kal08] (see also a different proof by Mathew [Mat20] and a revised original proof [KKM19]) showed that the 2-periodic version of the *Hodge-to-de Rham* spectral sequence
$$\mathrm{HH}_*(\mathcal{C}/k)[u^{\pm}] \Rightarrow \mathrm{HP}_*(\mathcal{C}/k)$$
degenerates at the E_2-page for any smooth proper stable ∞-category over field k of characteristic 0. For schemes over $\mathbb{C}$ the non-periodic version of this statement is one of the basic consequences of Hodge theory, for which no algebraic proof was known until the work of Faltings [Fal88] (see also Deligne and Illusie [DI87]).

This point of view is especially convenient for the purposes of studying more general geometric objects, such as derived schemes or stacks, to where, with the right definitions, the results automatically generalize. Moreover, the use of localizing invariants can be rewarding for studying abstract stable ∞-categories of non-geometric origin (such as the ∞-categories of motives or ∞-categories of modules over various ring spectra studied in topology). However, one must take into account the following limitations of the technique:

(1) Not all cohomology theories come from a localizing invariant. Important non-examples include "non-periodic" theories, such as motivic cohomology, and also "non-oriented" theories such as hermitian K-theory.

(2) Some geometric notions, such as nilpotent extension, and constructions such as (non-derived) pullbacks and pushouts do not have a resonable generalization in the plain categorical setting. Besides, the notions of regularity and smoothness in the context of stacks do not correspond to the categorical notions of regularity and smoothness. So it is not always possible to even formulate the pure categorical analogues of geometric statements.

With regards to the first issue, note that at least on smooth schemes over a field that admits resolution of singularities, all cohomology theories that are modules over the homotopy K-theory spectrum actually come from localizing invariants (see [Rob13, Corollary 1.16]), while the recent work [CDH$^+$20a, CDH$^+$20b, CDH$^+$20c] suggests a way to modify the formalism to include the hermitian K-theory and other non-oriented invariants into the picture by considering the so-called *Poincare ∞-categories* instead of stable ∞-categories.

The goal of this thesis is to address the second issue. We will shift our attention from the most general situation to a more restrictive setting of stable ∞-categories that are endowed with a *weight structure*.

Weight structures. A weight structure on a stable ∞-category is given by a choice of two subcategories with several properties reminiscent of the properties of a t-structure. The theory of weight structures was developed by Bondarko [Bon10] primarily for the purpose of studying stable ∞-categories of motivic origin (such as the ∞-category of Voevodsky motives $\mathbf{DM}(k)$). It has led to many advances in that area, however, weight structures are rarely mentioned in the context of non-commutative geometry. This has a reason – for a scheme X the stable ∞-category $\mathbf{Perf}_{\mathrm{X}}$ is rarely weighted, unless X is *affine*.

Our main proposal is to think of (boundedly) weighted stable ∞-categories as of noncommutative *affine* schemes. This provides a convenient language for talking about many geometric properties categorically while preserving the "coordinate-free" flavour of noncommutative geometry. In particular, in this context we will be able to talk about nilpotent extensions, regularity and connectivity in a way compatible with the corresponding notions for derived stacks. We will apply our abstract results about weight structures to the geometry of a large class of stacks and derived stacks (containing in particular all DM stacks). Our main applications are:

- A stacky analogue of the theorem of Dundas-Goodwillie-McCarthy,
- New cases of Blanc's lattice conjecture on the Chern character map $\mathrm{K}^{\mathrm{top}}(-) \to \mathrm{HP}(-)$,
- Pro-excision results for arbitrary localizing invariants on stacks,
- The Weibel's conjecture for stacks.

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The results of this thesis are presented in

Vladimir Sosnilo. Theorem of the heart in negative K-theory for weight structures. *Doc. Math.*, 24:2137–2158, 2019

Vladimir Sosnilo. Regularity of spectral stacks and discreteness of weight-hearts. *The Quarterly Journal of Mathematics*, 03 2021. haab017

Elden Elmanto and Vladimir Sosnilo. On nilpotent extensions of ∞-categories and the cyclotomic trace. *International Mathematics Research Notices*, 07 2021. rnab179

1. Summary of results

Here we briefly present the results of the thesis. We will follow the conventions presented below. These largely correspond to the conventions of many research papers in the area.

Conventions. We use extensively the language of ∞-categories. More precisely, by that we mean $(\infty, 1)$-categories and we follow the relatively standard notations of [Lur17b, Lur17a, Lur18].

(1) By a **category** we really do mean a 1-category and an ∞-category will be called as such.

(2) If $\mathcal{C}$ is an ∞-category we write $\mathcal{C}^{\simeq} \subset \mathcal{C}$ for its core aka maximal subgroupoid.

(3) If $\mathcal{C}$ is a stable ∞-category, then for each $x, y \in \mathcal{C}$ we denote its mapping spectrum by $\mathrm{maps}(x, y)$, while $\mathrm{Maps}(x, y)$ denotes the underlying mapping spaces so that $\Omega^\infty \mathrm{maps}(x, y) \simeq \mathrm{Maps}(x, y)$; the relationship between the notation $\mathrm{end}(x)$ and $\mathrm{End}(x)$ is analogous.

(4) If $\mathcal{C}$ is an ∞-category with finite coproducts, then we write

$$\mathrm{PSh}_\Sigma(\mathcal{C}) := \mathrm{Fun}^\times(\mathcal{C}^{\mathrm{op}}, \mathrm{Spc}),$$

for the ∞-category of functors which convert finite coproducts to products.

(5) For a (discrete) commutative ring R we identify R-linear pretriangulated dg-categories with R-linear stable ∞-categories (see [Hau15, Example 5.11]).

For the reader's convenience the main results of this thesis are surrounded by nice blue boxes.

First we need to introduce the notion of weight structures.

1.1. Weight structures.

We begin by introducing the main notion of this thesis—weight structures.

Definition 1.1.1. A **weight structure** on a stable ∞-category $\mathcal{C}$ consists of a pair of retract-closed full subcategories $(\mathcal{C}_{w\geqslant 0}, \mathcal{C}_{w\leqslant 0})$ such that:

(1) $\Sigma\mathcal{C}_{w\geqslant 0} \subset \mathcal{C}_{w\geqslant 0}$, $\Omega\mathcal{C}_{w\leqslant 0} \subset \mathcal{C}_{w\leqslant 0}$, we write

$$\mathcal{C}_{w\geqslant n} := \Sigma^n \mathcal{C}_{w\geqslant 0} \qquad \mathcal{C}_{w\leqslant k} := \Sigma^k \mathcal{C}_{w\leqslant 0}.$$

(2) if $x \in \mathcal{C}_{w\leqslant 0}, y \in \mathcal{C}_{w\geqslant 1}$ then

$$\pi_0 \mathrm{Maps}_{\mathcal{C}}(x, y) \simeq 0,$$

(3) for any object $x \in \mathcal{C}$ we have a cofiber sequence

$$x_{\leqslant 0} \to x \to x_{\geqslant 1},$$

where $x_{\leqslant 0} \in \mathcal{C}_{w\leqslant 0}$ and $x_{\geqslant 1} \in \mathcal{C}_{w\geqslant 1}$. We call these the **weight truncations** of x.

We say that the weight structure is **bounded** if

$$\mathcal{C} = \bigcup_n (\mathcal{C}_{w\geqslant -n} \cap \mathcal{C}_{w\leqslant n}).$$

A stable ∞-category $\mathcal{C}$ equipped with a (bounded) weight structure is called a (**boundedly**) **weighted stable ∞-category**. We call the subcategory spanned by $\mathcal{C}_{w\geqslant 0} \cap \mathcal{C}_{w\leqslant 0}$ the **heart** of w. We denote it by $\mathcal{C}^{\heartsuit_w}$.

Definition 1.1.2. Let $(\mathcal{C}, w), (\mathcal{D}, w')$ be weighted stable ∞-categories and let $f : \mathcal{C} \to \mathcal{D}$ be an exact functor of the underlying stable ∞-categories. We say that f is **weight exact** if $f(\mathcal{C}_{w\geqslant 0}) \subset \mathcal{D}_{w'\geqslant 0}$ and $f(\mathcal{C}_{w\leqslant 0}) \subset \mathcal{D}_{w'\leqslant 0}$. We denote by

$$\mathbf{WCat}_\infty^{\mathrm{st},b},$$

the ∞-category of small, boundedly weighted stable ∞-categories and weight exact functors.

Remark 1.1.3. Here we list some basic properties of weight structures.

(1) Each of the subcategories in the definition of a weight structure can be described as the corresponding orthogonal to the other class:

$$\mathcal{C}_{w\geqslant 0} = \{\mathrm{X} \in \mathcal{C} | \pi_0 \operatorname{Maps}(\mathrm{Y}, \mathrm{X}) = 0 \text{ for all } \mathrm{Y} \in \mathcal{C}_{w\leqslant -1}\}$$
$$\mathcal{C}_{w\leqslant 0} = \{\mathrm{X} \in \mathcal{C} | \pi_0 \operatorname{Maps}(\mathrm{X}, \mathrm{Y}) = 0 \text{ for all } \mathrm{Y} \in \mathcal{C}_{w\geqslant 1}\}$$

In other words, a weight structure gives rise to a *torsion theory* $(\mathcal{C}_{w\geqslant 0}, \mathcal{C}_{w\leqslant -1})$.

(2) Constructing a bounded weight structure is easy: given a full generating subcategory $\mathrm{N} \subset \mathcal{C}$ that is *connective*, i.e.

the mapping spectra $\operatorname{maps}(\mathrm{X}, \mathrm{Y})$ are connective for $\mathrm{X}, \mathrm{Y} \in \mathrm{N}$,

$\mathcal{C}_{w\geqslant 0}$ and $\mathcal{C}_{w\leqslant 0}$ defined as the full subcategories spanned by all finite colimits of N and finite limits of N, respectively, give a bounded weight structure on $\mathcal{C}$.

Example 1.1.4 (see also Example 3.1.4)**.** Let R be a connective ring spectrum. Then the ∞-category $\mathbf{Perf}_{\mathrm{R}}$ admits a bounded weight structure given by the subcategory of connective modules and the subcategory of modules with trivial positive R-cohomology. When $\mathrm{R} = \mathbb{S}$ the weight decompositions correspond precisely to the cellular filtrations. In particular, these are non-unique as they depend on a choice of a cellular structure.

Example 1.1.5. Let $\mathcal{A}$ be an additive category. Then the ∞-category of bounded (homological) chain complexes $\mathrm{Com}^b(\mathcal{A})$ admits a bounded weight structure with

$$\mathrm{M} \in \mathrm{Com}^b(\mathcal{A})_{w\geqslant 0} \iff \mathrm{M} \text{ is equivalent to a complex trivial in negative degrees,}$$
$$\mathrm{M} \in \mathrm{Com}^b(\mathcal{A})_{w\leqslant 0} \iff \mathrm{M} \text{ is equivalent to a complex trivial in positive degrees.}$$

In this case the weight decompositions correspond to brutal truncations.

We refer the reader to Section 3.3 for further information about weight structures.

1.2. **Reconstruction theorem for weighted categories.** Let $\mathcal{A}$ be an additive ∞-category. Consider the ∞-category

$$\mathrm{Fun}^\times(\mathcal{A}^{\mathrm{op}}, \mathrm{Spt})$$

of spectral-valued additive presheaves on $\mathcal{A}$. This is equivalent to the ∞-category of spectra in $\mathrm{PSh}_\Sigma(-)$ by [Lur18, Remark C.1.5.9] and hence is functorial in $\mathcal{A}$. Denote by $\mathcal{A}^{\mathrm{fin}}$ the minimal stable subcategory of $\mathrm{Fun}^\times(\mathcal{A}^{\mathrm{op}}, \mathrm{Spt})$ containing the Yoneda image of $\mathcal{A}$. Now the construction $\mathcal{A} \mapsto \mathcal{A}^{\mathrm{fin}}$ promotes to a functor

$$(-)^{\mathrm{fin}} : \mathbf{Cat}_\infty^{\mathrm{add}} \to \mathbf{Cat}_\infty^{\mathrm{st}},$$

where $\mathbf{Cat}_\infty^{\mathrm{add}}$ is the ∞-category of small additive ∞-categories and additive functors and $\mathbf{Cat}_\infty^{\mathrm{st}}$ is the ∞-category of small stable ∞-categories and exact functors. This is in fact the universal way to make a stable ∞-category out of an additive one. More precisely, $(-)^{\mathrm{fin}}$ is the left adjoint to the forgetful functor

$$\mathbf{Cat}_\infty^{\mathrm{st}} \to \mathbf{Cat}_\infty^{\mathrm{add}}.$$

There is a weight structure on $\mathcal{A}^{\mathrm{fin}}$ with $\mathcal{A}^{\mathrm{fin}}_{w\geqslant 0}$ and $\mathcal{A}^{\mathrm{fin}}_{w\leqslant 0}$ given respectively by finite colimits and finite limits of objects in the Yoneda image of $\mathcal{A}$. This gives us a functor

$$((-)^{\mathrm{fin}}, w) : \mathbf{Cat}_\infty^{\mathrm{add}} \to \mathbf{WCat}_\infty^{st,b}.$$

The main result of Section 3 is the following:

Theorem 1.2.1 (Theorem 3.2.7)**.** *The functor* $((-)^{\mathrm{fin}}, w)$ *is left adjoint to the functor taking a boundedly weighted stable* ∞*-category to its heart. Moreover, it restricts to an equivalence on the subcategories of idempotent complete* ∞*-categories.*

One corollary of Theorem 1.2.1 is the following universal property of the ∞-category of complexes. This is obtained by looking at the unit of the adjunction applied to the weighted stable ∞-category constructed in Example 1.1.5. This result can also be found in [BCKW19, Theorem 7.56].

Corollary 1.2.2 (Corollary 3.2.8)**.** *For an additive category* $\mathcal{A}$ *there is an equivalence of (weighted) stable* ∞*-categories*

$$\mathcal{A}^{\mathrm{fin}} \simeq \mathrm{Com}^b(\mathcal{A}).$$

In other words, $\mathrm{Com}^b(\mathcal{A})$ *is the free stable* ∞*-category on an additive category.*

1.2.3. *Weight structures on stacks.* The main source of examples of weight structures for this thesis is the following theorem. The case of G being the trivial group scheme recovers Example 1.1.4 for $\mathbb{E}_\infty$-k-algebras.

Theorem 1.2.4 (Theorem 3.3.13)**.** *Let* k *be a commutative ring and* G *be an embeddable linearly reductive group scheme over* k*.*

(1) Let R *be a connective* $\mathbb{E}_\infty$*-*k*-algebra endowed with an action of* G*. There is a bounded weight structure on* $\mathbf{Perf}_{[\mathrm{Spec}\,\mathrm{R}/\mathrm{G}]}$ *whose heart is the full subcategory generated under retracts by the set*

$$\mathrm{N} := \{p^*\mathcal{E} : \mathcal{E} \text{ is a finite } \mathrm{G}\text{-equivariant } k\text{-module } \mathcal{E}\},$$

where $p : [\mathrm{Spec}\,\mathrm{R}/\mathrm{G}] \to \mathrm{BG}$ *is the structure morphism.*

(2) Let $\mathrm{R} \xrightarrow{f} \mathrm{S}$ *be a* G*-equivariant morphism of connective* $\mathbb{E}_\infty$*-*k*-algebras endowed with an action of* G*. Then the base change functor* $\mathbf{Perf}_{[\mathrm{Spec}\,\mathrm{R}/\mathrm{G}]} \to \mathbf{Perf}_{[\mathrm{Spec}\,\mathrm{S}/\mathrm{G}]}$ *is weight exact.*

1.2.5. *Applications.* Theorem 1.2.1 gives weighted ∞-categories a lot of flexibility. Now both the constructions of the theory of stable ∞-categories (such as Verdier localization) and of the theory of additive ∞-categories are available for weighted stable ∞-categories. One useful example of a construction coming from the world of additive ∞-categories is the following.

Example 1.2.6. Given an idempotent complete boundedly weighted ∞-category $\mathcal{C}$, we have an equivalence $(\mathcal{C}^{\heartsuit_w})^{\mathrm{fin}} \to \mathcal{C}$. In particular we have a canonical functor

$$\mathcal{C} \simeq (\mathcal{C}^{\heartsuit_w})^{\mathrm{fin}} \to (\mathrm{h}\mathcal{C}^{\heartsuit_w})^{\mathrm{fin}} \simeq \mathrm{Com}^b(\mathrm{h}\mathcal{C}^{\heartsuit_w}),$$

where the last equivalence is given by Corollary 1.2.2. In Bondarko's terminology this is called the *strong weight complex functor*[2]. This example is crucial for most applications of the current thesis (and, arguably, of the whole theory).

Example 1.2.7 (see Section 3.3.1)**.** Let k be a field of exponential characteristic p and R be a ring where p is invertible. We consider the ∞-category of Voevodsky motives $\mathbf{DM}(k;\mathrm{R})$ and its full subcategory of compact objects $\mathbf{DM}_{\mathrm{gm}}(k;\mathrm{R})$. The full subcategory of $\mathbf{DM}_{\mathrm{gm}}(k;\mathrm{R})$ spanned by motives of smooth projectives is connective and generating. Therefore by Remark 1.1.3(2) there is a weight structure on $\mathbf{DM}_{\mathrm{gm}}(k;\mathrm{R})$. We call this the *Chow weight structure.* The homotopy category of its heart is equivalent to the additive category of Chow motives $\mathbf{Chow}(k;\mathrm{R})$.

[2]An analogue of this functor can be defined in the triangulated context for any triangulated category with a weight structure $(\mathcal{C}, w)$. Its codomain is a certain **non-triangulated** category which is a quotient of $h\mathrm{Com}^b(\mathcal{C}^w)$ by a square-zero ideal.

Combining the two examples we obtain a functor

$$\mathbf{DM}_{\mathrm{gm}}(k;\mathrm{R}) \to \mathrm{Com}^b(\mathbf{Chow}(k;\mathrm{R})).$$

Corollary 1.2.8 (Corollary 5.3.12 and Remark 5.3.13)**.** *For any boundedly weighted stable ∞-category $(\mathcal{C}, w)$ the functor*

$$\mathcal{C} \to \mathrm{Com}^b(h\mathcal{C}^{\heartsuit_w})$$

induces an equivalence in negative K*-theory. In case of $\mathcal{C} = \mathbf{DM}_{\mathrm{gm}}(k;\mathrm{R})$ this, in combination with* [AGH19]*, implies that the additive* K*-theory groups*

$$\mathrm{K}^{\Sigma}_{-n}(\mathbf{Chow}(k;\mathrm{R}))$$

for $n > 0$ vanish whenever $\mathbf{DM}_{\mathrm{gm}}(k;\mathrm{R})$ admits a motivic t-structure with noetherian heart.

1.3. **Regular stacks.** In Section 4 we study regular derived stacks. This section provides the first example of a connection between the theories of weight structures and the theory of derived stacks. This section is mostly independent of the rest of the thesis.

Recall that a commutative noetherian ring R is regular if and only if any coherent module over R has a finite projective resolution. This condition is equivalently saying that there holds an inclusion

$$\mathbf{Coh}_{\mathrm{R}} \subseteq \mathbf{Perf}_{\mathrm{R}}$$

of stable subcategories of $\mathbf{Mod}_{\mathrm{R}}$. This observation can be used to define regularity in more general contexts. For example, regularity of connective ring spectra was introduced in [BL14]. More generally, we define:

Definition 1.3.1. A noetherian connective spectral stack X is called **regular** if and only if any coherent module over X is perfect[3].

We show in Section 4 that this notion is somewhat well-behaved, in particular we prove the following:

Theorem 1.3.2. *Let* X *be a quotient stack* [Spec(R)/G] *where* R *is an $\mathcal{E}_\infty$-k-algebra spectrum for a field k and* G *is a smooth linearly reductive group scheme over k. Assume that $\pi_0(\mathrm{R})$ is of finite type over k and the graded ring $\pi_*(\mathrm{R})$ is graded noetherian. Then the following statements are valid:*

(1) X *is regular if and only if* R *is regular.*
(2) X *is discrete if it is regular and* R *has finitely many non-zero homotopy groups.*

Remark 1.3.3. Part (1) of Theorem 1.3.2 is saying that the regularity property is local in the smooth topology on such quotient stacks (for $\mathbb{E}_\infty$-ring spectra this statement follows from [Lur18, Proposition 2.8.4.2]).

Part (2) of Theorem 1.3.2 is an instance of the principle that there are no nilpotent functions on a non-singular stack. Indeed, elements of the higher homotopy groups of a bounded ring spectrum R are nilpotent in the graded homotopy ring, and thus may be interpreted as nilpotents on X.

1.3.4. *Regular weight structures.* Unfortunately, Definition 1.3.1 is not purely noncommutative in the sense described in the introduction, since it depends on $\mathbf{Perf}_{\mathrm{R}}$ *as well as* a subcategory

$$\mathbf{Coh}_{\mathrm{R}} \subset \mathrm{Ind}(\mathbf{Perf}_{\mathrm{R}}) = \mathbf{Mod}_{\mathrm{R}},$$

[3]For X being the spectrum of a connective $\mathbb{E}_\infty$-ring spectrum this definition slightly differs from the one given by Barwick and Lawson; they additionally require $\pi_0(\mathrm{X})$ to be a regular classical scheme, otherwise calling it *almost regular*.

and not just on the stable ∞-category $\mathbf{Perf}_{\mathrm{R}}$. There does exist a purely noncommutative criterion for regularity of classical rings [Orl16], [Nee92] but this does not give a very interesting notion for stacks (for example, $\mathbf{Perf}_{\mathrm{X}}$ is never generated by a single object if X has a point with infinite stabilizer group and in particular, there is no analogue of Theorem 1.3.2).

However, $\mathbf{Coh}_{\mathrm{R}}$ can be easily recovered from the canonical weight structure on $\mathbf{Perf}_{\mathrm{R}}$ (Example 1.1.4) using the following result of Bondarko. We expect this point of view to be helpful for studying localizing invariants of regular spectral stacks and, generally, for better understanding of this property.

Theorem 1.3.5 (Theorem 0.1 in [Bon18])**.** *Let $\mathcal{C}$ be a stable ∞-category with a weight structure, then there is a canonical weight structure w and a t-structure t_w on* $\mathrm{Ind}(\mathcal{C})$ *with*

$$\mathrm{Ind}(\mathcal{C})_{w\geqslant 0} = \mathrm{Ind}(\mathcal{C})_{t_w\geqslant 0} = \mathrm{Ind}(\mathcal{C}_{w\geqslant 0}).$$

In this language $\mathbf{Coh}_{\mathrm{R}}$ can be defined as the minimal subcategory of $\mathrm{Ind}(\mathbf{Perf}_{\mathrm{R}})$ generated by $\pi_i^{t_w}(\mathrm{P})$ for all perfect complexes P under finite colimits and shifts. This leads to the following definition.

Definition 1.3.6. A boundedly weighted stable ∞-category $\mathcal{C}$ is called **regular** if the canonical t-structure on $\mathrm{Ind}(\mathcal{C})$ given by Theorem 1.3.5 restricts to $\mathcal{C}$.

Remark 1.3.7. Let X be either a quotient stack $[\operatorname{Spec}\mathrm{R}/\mathrm{G}]$ as in Theorem 1.2.4(1) or Spec of a connective $\mathbb{E}_\infty$-ring spectrum, then $\mathbf{Perf}_{\mathrm{X}}$ is regular if and only if X is regular.

1.4. **Theorem of Dundas-Goodwillie-McCarthy for weight structures.** The celebrated Dundas-Goodwillie-McCarthy (DGM) theorem is an incredibly useful tool for studying K-theory and is the starting point for trace methods.

Theorem 1.4.1 (Dundas-Goodwillie-McCarthy)**.** *Let* $\mathrm{A} \to \mathrm{B}$ *be a morphism of connective* $\mathbb{E}_1$*-ring spectra such that the map* $\pi_0(\mathrm{A}) \to \pi_0(\mathrm{B})$ *is surjective and the kernel is a nilpotent ideal of* $\pi_0(\mathrm{A})$*. Then the trace map from algebraic* K*-theory to topological cyclic homology (*TC*)*

$$\mathrm{tr} : \mathrm{K} \to \mathrm{TC}$$

induces a cartesian square

$$\begin{array}{ccc} \mathrm{K(A)} & \longrightarrow & \mathrm{TC(A)} \\ \downarrow & & \downarrow \\ \mathrm{K(B)} & \longrightarrow & \mathrm{TC(B)}. \end{array}$$

Both K-theory and TC can be extended to localizing invariants of stable ∞-categories, and the trace map is a map of localizing invariants. So it is not unnatural to expect a version of the DGM-theorem for certain functors of stable ∞-categories. However, as we warned the reader in the introduction, it is sometimes problematic to translate statements directly into the noncommutative setting. In this particular case it is not clear what should replace the notions

- connectivity of ring spectra,
- π_0-surjectivity of maps,
- nilpotence of the kernel ideal of a map.

This translation becomes transparent in the presence of weight structures.

1.4.2. *Nilpotent extensions of weighted categories.* In Section 5 we introduce a notion of a **nilpotent extension** of additive ∞-categories and of weighted stable ∞-categories (see Definition 5.1.1). In the additive case the notion is a straightforward generalization of nilpotent extension of rings, i.e. a surjective map whose kernel is a nilpotent ideal. This is then translated directly into the weighted case via the equivalence of Theorem 1.2.1.

The main result of this section is the following theorem.

Theorem 1.4.3 (Theorem 5.4.4)**.** *Let* E *be a localizing invariant that is truncating (i.e.* $\mathrm{E}(\mathrm{R}) = \mathrm{E}(\pi_0(\mathrm{R}))$ *for all connective ring spectra). Let* $f : (\mathcal{A}, w) \to (\mathcal{B}, w')$ *be a nilpotent extension of boundedly weighted stable* ∞*-categories. Then we have an induced equivalence of spectra*

$$\mathrm{E}(\mathcal{A}) \to \mathrm{E}(\mathcal{B}).$$

Remark 1.4.4. The result is easy for those truncating localizing invariants that commute with filtered colimits. Indeed, one could present $\mathcal{A}$ and $\mathcal{B}$ as filtered colimits of ∞-categories of perfect complexes over endomorphism rings of objects in the hearts $\mathcal{A}^{\heartsuit_w}$ and $\mathcal{B}^{\heartsuit_w}$. These endomorphism rings are connective by the orthogonality axioms of weight structures. In fact it is easy to see that a nilpotent extension between these boundedly weighted stable ∞-categories of perfect complexes comes from nilpotent extensions of these connective ring spectra. So now the result follows formally from the assumptions and [LT19, Corollary 3.5].

However, in many cases of interest E does not commute with filtered colimits. One example is the next corollary which we think of as the *categorical version of DGM-theorem.*

Corollary 1.4.5 (Corollary 5.4.6)**.** *Let* $f : (\mathcal{A}, w) \to (\mathcal{B}, w')$ *be a nilpotent extension of boundedly weighted stable* ∞*-categories. Then we have an induced cartesian square of spectra:*

$$\begin{array}{ccc} \mathrm{K}(\mathcal{A}) & \longrightarrow & \mathrm{TC}(\mathcal{A}) \\ \downarrow & & \downarrow \\ \mathrm{K}(\mathcal{B}) & \longrightarrow & \mathrm{TC}(\mathcal{B}). \end{array}$$

We hope that this Corollary will be a useful tool for computing K-theory of various stable ∞-categories. As one example, we apply it to the Chow weight structure on the ∞-category of Voevodsky motives.

Corollary 1.4.6 (Example 5.4.9)**.** *Let* k *be a field of exponential characteristic* e *and* R *a ring of coefficients such that* e *is invertible in* R*. Then the square*

$$\begin{array}{ccc} \mathrm{K}(\mathbf{DM}_{\mathrm{gm}}(k,\mathrm{R})) & \longrightarrow & \mathrm{TC}(\mathbf{DM}_{\mathrm{gm}}(k,\mathrm{R})) \\ \downarrow & & \downarrow \\ \mathrm{K}^{\Sigma}(\mathbf{Chow}(k,\mathrm{R})) & \longrightarrow & \mathrm{TC}^{\Sigma}(\mathbf{Chow}(k,\mathrm{R})) \end{array}$$

is cartesian.

Now fix a base discrete commutative ring k and let G be a linearly reductive, embeddable group scheme over k. If E is a localizing invariant and R is a connective $\mathbb{E}_\infty$-k-algebra with a G-action, we set

$$\mathrm{E}^{\mathrm{G}}(\mathrm{R}) := \mathrm{E}(\mathbf{Perf}_{[\mathrm{Spec}\,\mathrm{R}/\mathrm{G}]}),$$

which is a version of G-equivariant E-theory.

Corollary 1.4.7 (Equivariant DGM-theorem; Corollary 5.4.12)**.** *Let* $\mathrm{R} \to \mathrm{S}$ *be a* G*-equivariant map of connective* $\mathbb{E}_\infty$*-*k*-algebras. Then the square*

$$\begin{array}{ccc} \mathrm{K}^{\mathrm{G}}(\mathrm{R}) & \longrightarrow & \mathrm{TC}^{\mathrm{G}}(\mathrm{R}) \\ \downarrow & & \downarrow \\ \mathrm{K}^{\mathrm{G}}(\mathrm{S}) & \longrightarrow & \mathrm{TC}^{\mathrm{G}}(\mathrm{S}) \end{array}$$

is cartesian.

1.5. **Categorical Milnor excision and pro-excision.** Another useful technique in algebraic K-theory is given by various descent properties. This often can be presented in the form of having Mayer-Vietoris type long exact sequences available for computation. In particular, it has been known for a long time that Milnor squares and abstract blow-up squares of schemes induce long exact sequences in K-theory in some special situations (see [Mor18]). More generally, K-theory was shown to satisfy the so-called pro-excision and pro-cdh-descent properties in [KST17] (see also [LT19]).

We introduce a notion of a *Milnor square* of stable ∞-categories (Definition 6.2.1) and prove the following result which is ideologically a reinterpretation of the proof of [LT19]:

Theorem 1.5.1 (Theorem 6.2.4)**.** *Any localizing invariant sends a Milnor square of stable* ∞*-categories into a pullback square.*

Possibly somewhat confusingly, Milnor squares correspond not to Milnor squares of schemes but rather to their derived analogue. However, in geometry the difference disappears after passing to the ind-system of nilpotent thickenings of the closed embeddings. This leads us to introducing a pro-version of Milnor square (Definition 6.4.2) and proving the following:

Theorem 1.5.2 (Theorem 6.4.5)**.** *Assume given a commutative square of cofiltered diagrams of boundedly weighted stable* ∞*-categories*

$$\begin{array}{ccc} \{\mathcal{A}_\lambda\} & \xrightarrow{\{f_\lambda\}} & \{\mathcal{B}_\lambda\} \\ \big\downarrow{\scriptstyle \{p_\lambda\}} & & \big\downarrow{\scriptstyle \{q_\lambda\}} \\ \{\mathcal{A}'_\lambda\} & \xrightarrow{\{g_\lambda\}} & \{\mathcal{B}'_\lambda\}. \end{array} \tag{1.5.3}$$

which is a pro-Milnor square and such that for any λ *there exists an integer* n *such that* $\mathcal{A}_\lambda^\heartsuit$, $\mathcal{A'}_\lambda^\heartsuit$, $\mathcal{B}_\lambda^\heartsuit$ *and* $\mathcal{B}_\lambda^{'\heartsuit}$ *are* n*-categories. Then any localizing invariant sends it to a pullback square in pro-spectra.*

This isn't a straightforward consequence of Theorem 1.5.1. A crucial step here, which may be of independent interest, is Corollary 6.3.11 below. This gives a criterion for a functor of pro-objects of weighted stable ∞-categories to be a pro-equivalence in terms of what it induces on mapping spaces. This step uses weight structures substantially and at the moment we don't know how to prove or even formulate an analogue of this statement for more general stable ∞-categories.

1.6. **Applications to stacks.** After doing all the categorical work we can finally pick all the fruits. The last main ingredient is the local structure theorem given by the results of [AHR19] and [AHHLR]. This result says that a large class of stacks (which we call *ANS stacks*) admit Nisnevich covers by stacks of the form $[\operatorname{Spec} \mathrm{R}/\mathrm{G}]$. Now we can use all the categorical machinery presented in previous sections as well as Theorem 1.2.4.

1.6.1. *Pro-cdh-descent and Weibel's conjecture.* Recall that an **abstract blow-up square** of *noetherian* algebraic stacks

$$\begin{array}{ccc} \mathrm{Z} & \longrightarrow & \tilde{\mathrm{X}} \\ \downarrow & & \downarrow p \\ \mathrm{Y} & \xrightarrow{i} & \mathrm{X}, \end{array}$$

is a commutative square with p proper representable and i closed immersion such that p induces an isomorphism $\tilde{\mathrm{X}} - \mathrm{Z} \to \mathrm{X} - \mathrm{Y}$. Denote by Z_n and Y_n the n-th nilpotent thickenings of Z in $\tilde{\mathrm{X}}$ and Y in X, respectively.

We say that a derived stack is an ANS stack if it has affine diagonal and linearly reductive almost multiplicative stabilizers.

We prove the following theorem generalizing the main result of [KST17] (see also [LT19]) to stacks.

Theorem 1.6.2 (Theorem 7.3.1)**.** *Assume* X *is an ANS stack. Then for any localizing invariant* E *the square*

$$\begin{array}{ccc} \mathrm{E}(\mathrm{X}) & \longrightarrow & \text{“}\lim\text{”}\,\mathrm{E}(\mathrm{Y}_n) \\ \downarrow & & \downarrow \\ \mathrm{E}(\tilde{\mathrm{X}}) & \longrightarrow & \text{“}\lim\text{”}\,\mathrm{E}(\mathrm{E}_n) \end{array}$$

is a pullback of pro-spectra.

This has the following corollary which in the case $\mathrm{E} = \mathrm{KH}$ has been obtained in [HK19].

Corollary 1.6.3 (Corollary 7.7.4)**.** *Assume* X *is an ANS algebraic stack and* E *is a truncating invariant. Then the square*

$$\begin{array}{ccc} \mathrm{E}(\mathrm{X}) & \longrightarrow & \mathrm{E}(\mathrm{Y}) \\ \downarrow & & \downarrow \\ \mathrm{E}(\tilde{\mathrm{X}}) & \longrightarrow & \mathrm{E}(\mathrm{Z}) \end{array}$$

is a pullback of spectra. Therefore truncating invariants of ANS algebraic stacks satisfy cdh descent.

As another consequence of that, we prove an appropriate version of Weibel's conjecture on the vanishing of negative K-theory, similarly to how it is done in [HK19] and [KS16].

Theorem 1.6.4 (Theorem 7.3.3)**.** *Let* X *be a noetherian ANS stack of covering dimension* d *(see Definition 7.2.1). Then* $\mathrm{K}_{-n}(\mathrm{X})$ *vanishes for all* $n > d$*.*

1.6.5. *Lattice conjecture.* Combining our results with the results of Halpern-Leinster and Pomerleano and of Konovalov we are able to obtain new cases of the lattice conjecture of Blanc:

Theorem 1.6.6 (Theorem 7.8.3)**.** *Let* X *be a derived stack over* $\mathbb{C}$ *satisfying any of the following hypotheses:*

(1) it is of the form [Y/G] *where* Y *is a derived, affine* G*-scheme over* $\mathbb{C}$*,* G *is a reductive* $\mathbb{C}$*-group scheme such that* Y^{cl} *is a smooth* $\mathbb{C}$*-scheme;*
(2) it is ANS derived stack such that X^{cl} *is a smooth* $\mathbb{C}$*-stack.*

Then, the complexified Chern character map

$$\mathrm{Ch} \otimes \mathbb{C} : \mathrm{K}^{\mathrm{top}}(\mathrm{X}) \otimes \mathbb{C} \to \mathrm{HP}(\mathrm{X})$$

from the topological K-theory (defined as a localizing invariant by Blanc) to the periodic cyclic homology is an equivalence.

2. On additive ∞-categories

In this section, we develop more thoroughly some aspects of the theory of additive ∞-categories, such as their localization and idempotent completion. We keep in mind the analogs of these concepts for stable ∞-categories and how they interact with localizing invariants. A result which might be of independent interest is a Schwede-Shipley-style recognition principle for additive ∞-categories, given in Theorem 5.3.1. This result feeds into the proof of our main result. We finally explain how Bondarko's theory of weight structures builds a "bridge" between additive and stable ∞-categories.

2.1. **Basics.** Recall that an additive category is, in particular, enriched in the category of abelian groups. Hence, if $x \in \mathrm{Obj}(\mathcal{A})$ then $\mathrm{Hom}(x, x)$ is naturally an associative ring with $\circ$ acting as the multiplication. From this point of view, it is natural to view an additive category as the generalization of an associative ring. In higher algebra, the analog of an additive category is an additive ∞-category [GGN15], [Lur18, Appendix C.1.5].

Definition 2.1.1. A **semi-additive ∞-category** (also often called **preadditive ∞-category**) $\mathcal{A}$ is a *pointed* ∞-category with finite products and coproducts such that for any pair of objects x, y, the canonical map

$$x \sqcup y \to x \times y,$$

is an equivalence. We write this object as the **sum** $x \oplus y$. The sum admits a **shear map**

$$s = (\pi_1, \nabla) : x \oplus x \to x \oplus x,$$

where π_1 is the first projection and ∇ is the fold map. If this map is an equivalence for all $x \in \mathrm{Obj}(\mathcal{A})$, then we say that $\mathcal{A}$ is an **additive ∞-category**. An **additive functor** of additive ∞-categories is a functor $\mathrm{F} : \mathcal{A} \to \mathcal{B}$ that preserves zero objects and all sums.

We refer to [GGN15, Section 2] for a more extensive discussion and the following result which is [GGN15, Proposition 2.8]:

Proposition 2.1.2. *Let $\mathcal{C}$ be an ∞-category with finite coproducts and products. Then the following are equivalent:*

1. *the ∞-category $\mathcal{C}$ is additive,*
2. *the homotopy category $h\mathcal{C}$ is additive,*
3. *the forgetful functor*
$$\mathrm{Gp}_{\mathbb{E}_\infty}(\mathcal{C}) \to \mathcal{C}$$
is an equivalence.

Our first goal is to develop a theory of exact sequences of additive ∞-categories. We will consider the non-full subcategory

$$\mathbf{Cat}_\infty^{\mathrm{add}} \subset \mathbf{Cat}_\infty$$

spanned by *small* additive ∞-categories and additive functors between them.

2.2. **Verdier quotients.** To develop the theory of exact sequences of additive ∞-categories we will follow the case of stable ∞-categories analyzed in [BGT13, Section 5] and [NS18, Chapter I]. In the 1-categorical context, this was studied by the author and Bondarko in [BS18b].

Before we proceed, let us gather the necessary ingredients. First, for a small ∞-category $\mathcal{C}$ and a collection of arrows $\mathrm{W} \subset \mathcal{C}$ we can form the ∞-category $\mathcal{C}[\mathrm{W}^{-1}]$ equipped with a functor $\gamma : \mathcal{C} \to \mathcal{C}[\mathrm{W}^{-1}]$ which is the universal ∞-category under $\mathcal{C}$ where all elements in W are invertible (more precisely, see the universal property [Cis19, Definition 7.1.2]). This localization exists and is unique up to equivalence of ∞-categories under $\mathcal{C}$ [Cis19, Proposition 7.1.3]. Moreover, the functor can be chosen to be identity on objects [Cis19, Remark 7.1.4]. Furthermore, the functor $h(\mathcal{C}) \to h(\mathcal{C}[W^{-1}])$ witnesses $h(\mathcal{C}[W^{-1}])$ as a localization of the category $h(\mathcal{C})$ in the sense of ordinary category theory (as reviewed, for example, in [Cis19, Definition 2.2.8]).

Second, we need a passage from "small categories" to "big categories." For motivation, let us recall that in the context of stable ∞-categories what one might mean by this is the passage to ind-completion. For $\mathcal{C}, \mathcal{D}$ stable ∞-categories, we write $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{D})$ for the ∞-category of functors which are **exact**, i.e., preserve finite limits and colimits. If $\mathcal{C}$ is a small stable ∞-category, then the Yoneda functor

$$よ : \mathcal{C} \to \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Spt}),$$

identifies the codomain with $\mathrm{Ind}(\mathcal{C})$; see, for example, [BGT13, Proposition 3.2]. The upshot of going to $\mathrm{Ind}(\mathcal{C})$ is that the resulting ∞-category is a stable presentable ∞-category. Therefore localizations of this ∞-category can often be constructed via adjoint functor theorems [Lur17b, Corollary 5.5.2.9]. The theory of localization sequences in algebraic K-theory as pioneered by Thomason [TT90], Neeman [Nee92], and later in [BGT13] is developed by passing to large ∞-categories first before going back to small categories by taking compact objects and the functor

$$\mathcal{C} \to \mathrm{Ind}(\mathcal{C})^{\omega},$$

then witnesses the idempotent completion of $\mathcal{C}$ [Lur17b, Lemma 5.4.2.4].

For additive ∞-categories, by passage from the "small categories" to "big categories" we mean the passage to Quillen's nonabelian derived categories through which the Yoneda embedding factors:

$$よ : \mathcal{C} \to \mathrm{PSh}_{\Sigma}(\mathcal{C}).$$

In this case, $\mathrm{PSh}_{\Sigma}(\mathcal{C})$ is a prestable presentable ∞-category. The basic properties of this construction are summarized in the next lemma.

Lemma 2.2.1. *Let $\mathcal{C}$ be a small ∞-category with finite coproducts. Consider the Yoneda functor*

$$よ : \mathcal{C} \to \mathrm{PSh}_{\Sigma}(\mathcal{C}) \subset \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Spc}).$$

Then:

(1) the ∞-category $\mathcal{C}$ is additive if and only if $\mathrm{PSh}_{\Sigma}(\mathcal{C})$ is prestable.

(2) in this case, we have an equivalence:

$$\mathrm{PSh}_{\Sigma}(\mathcal{C}) \simeq \mathrm{Fun}^{\times}(\mathcal{C}^{\mathrm{op}}, \mathrm{Spt}_{\geqslant 0}), \textit{and}$$

the ∞-category $\mathrm{PSh}_{\Sigma}(\mathcal{C})$ is a presentable prestable ∞-category.

Proof. See [Lur18, Proposition C.1.5.7 and Remark C.1.5.9]. □

One consequence of this is an enhancement of the Yoneda functor.

Corollary 2.2.2. *Let $\mathcal{A}$ be an additive ∞-category. The Yoneda functor $\mathcal{A} \to \mathrm{PSh}_{\Sigma}(\mathcal{A})$ canonically promotes to a functor*

$$\mathcal{A} \to \mathrm{Fun}^{\times}(\mathcal{A}^{\mathrm{op}}, \mathrm{Spt}),$$

which is fully faithful.

Proof. The functor is given by:

$$\mathcal{A} \to \mathrm{PSh}_{\Sigma}(\mathcal{A}) \simeq \mathrm{Fun}^{\times}(\mathcal{A}^{\mathrm{op}}, \mathrm{Spt}_{\geqslant 0}) \hookrightarrow \mathrm{Fun}^{\times}(\mathcal{A}^{\mathrm{op}}, \mathrm{Spt}),$$

where the equivalence is given by Lemma 2.2.1(2) and the second functor is induced by the inclusion of connective spectra into all spectra. The functor in question is therefore a composite of two fully faithful functors. □

We use notation

$$\widehat{\mathcal{A}} := \mathrm{Fun}^{\times}(\mathcal{A}^{\mathrm{op}}, \mathrm{Spt}),$$

and call the resulting ∞-category the ∞-category of **additive spectral presheaves**.

We now address the theory of Verdier quotients in the context of additive ∞-categories using the nonabelian derived category in analogy with the development of Verdier quotients for stable ∞-categories using Ind-completions as in [BGT13, Section 5]; see also [NS18, Section I.3].

From hereon, we adopt the following notation. If $\mathcal{A}$ is an additive ∞-category and x is an object of $\mathcal{A}$, then the functor represented by x:

$$\mathrm{Maps}_{\mathcal{A}}(-, x) : \mathcal{A}^{\mathrm{op}} \to \mathrm{Spc},$$

is a product-preserving functor and hence an object of $\mathrm{PSh}_{\Sigma}(\mathcal{A})$. By Lemma 2.2.1(2), this functor canonically upgrades to a functor into connective spectra. We denote this functor by

$$\mathbf{Maps}_{\mathcal{A}}(-, x) : \mathcal{A}^{\mathrm{op}} \to \mathrm{Spt}_{\geqslant 0},$$

and set

$$\mathbf{End}_{\mathcal{A}}(x) := \mathbf{Maps}_{\mathcal{A}}(x, x).$$

In what follows also note that cofibers of maps of connective spectra can be computed in spectra.

Theorem 2.2.3. *Let $\mathcal{A}$ be a small additive ∞-category and $\mathcal{B} \subset \mathcal{A}$ be an additive full subcategory. Then*

(1) Define

$$\mathrm{W} := \{f \oplus g \colon b \oplus a \to b' \oplus a' \text{ where } f \colon b \to b' \in \mathcal{B} \text{ and } g : a \to a' \text{ is invertible}\}.$$

$\mathcal{A}/\mathcal{B} := \mathcal{A}[\mathrm{W}^{-1}]$ is an additive ∞-category equipped with an additive functor

$$\overline{(-)} : \mathcal{A} \to \mathcal{A}/\mathcal{B}$$

which is a bijection on the sets of objects.

(2) If $\mathcal{E}$ is another additive ∞-category, then $\overline{(-)} : \mathcal{A} \to \mathcal{A}/\mathcal{B}$ induces an equivalence between $\mathrm{Fun}^{\times}(\mathcal{A}/\mathcal{B}, \mathcal{E})$ and the full subcategory of $\mathrm{Fun}^{\times}(\mathcal{A}, \mathcal{E})$ consisting of those additive functors $\mathrm{G} : \mathcal{A} \to \mathcal{E}$ satisfying $\mathrm{G}|_{\mathcal{B}} \simeq 0$.

(3) We have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\overline{(-)}} & \mathcal{A}/\mathcal{B} \\ \downarrow & & \downarrow \\ \mathrm{PSh}_{\Sigma}(\mathcal{A}) & \xrightarrow{\mathrm{L}} & \mathrm{PSh}_{\Sigma}(\mathcal{A}/\mathcal{B}), \end{array}$$

where the bottom horizontal functor is left adjoint to the functor

$$\mathrm{res}_{\mathrm{can}} : \mathrm{PSh}_{\Sigma}(\mathcal{A}/\mathcal{B}) \to \mathrm{PSh}_{\Sigma}(\mathcal{A}) \qquad \mathrm{F} \mapsto \mathrm{F} \circ \overline{(-)}.$$

(4) If $\mathcal{B}$ is generated under sums and retracts by a single object z, then for any $\mathrm{F} \in \mathrm{PSh}_{\Sigma}(\mathcal{A})$ we have:

$$\mathrm{LF}(-) \simeq \mathrm{Cofib}(\mathrm{F}(z) \otimes_{\mathbf{End}_{\mathcal{A}}(z)} \mathbf{Maps}_{\mathcal{A}}(-, z) \to \mathrm{F}(-)).$$

(5) In general for any $\mathrm{F} \in \mathrm{PSh}_{\Sigma}(\mathcal{A})$ there is an equivalence:

$$\mathrm{LF}(-) \simeq \mathrm{Cofib}(\operatorname*{colim}_{z_1, \cdots, z_n \in \mathcal{B}} \mathrm{F}(\bigoplus_{i=1}^{n} z_i) \otimes_{\mathbf{End}_{\mathcal{A}}(\bigoplus_{i=1}^{n} z_i)} \mathbf{Maps}_{\mathcal{A}}(-, \bigoplus_{i=1}^{n} z_i) \to \mathrm{F}(-)),$$

where the colimit is taken over the poset of finite subsets $\{z_1, \cdots, z_n\}$ of objects of $\mathcal{B}$.

Proof. To prove assertion (1) it suffices to show that the the functor on homotopy categories

$$h(\mathcal{A}) \to h(\mathcal{A}[\mathrm{W}^{-1}])$$

is an additive functor of additive categories by Proposition 2.1.2. This follows from [BS18b, Proposition 2.2.4].

By the universal property of the localization, $\overline{(-)} : \mathcal{A} \to \mathcal{A}/\mathcal{B}$ induces an equivalence between $\mathrm{Fun}^{\times}(\mathcal{A}/\mathcal{B}, \mathcal{E})$ and the full subcategory of $\mathrm{Fun}^{\times}(\mathcal{A}, \mathcal{E})$ consisting of those additive functors $\mathrm{G} : \mathcal{A} \to \mathcal{E}$ for which $\mathrm{G}(w)$ is an equivalence for all $w \in \mathrm{W}$. By definition of W and additivity of G the latter condition is equivalent to $\mathrm{G}(g)$ being an equivalence for any $g \colon b \to b' \in \mathcal{B}$ which is also equivalent to: $\mathrm{G}(b) \simeq 0$ for all $b \in \mathcal{B}$. This proves (2).

To prove assertion (3), we start with the following general observations about an additive functor of ∞-categories $\pi : \mathcal{A}_0 \to \mathcal{A}_1$. The restriction of a product-preserving functor $(\mathcal{A}_1)^{\mathrm{op}} \to \mathrm{Spc}$ along π remains product-preserving since π is additive. Therefore we have a commutative diagram

$$\begin{array}{ccc} \mathrm{PSh}_{\Sigma}(\mathcal{A}_1) & \xrightarrow{\pi^*} & \mathrm{PSh}_{\Sigma}(\mathcal{A}_0) \\ \downarrow & & \downarrow \\ \mathrm{PSh}(\mathcal{A}_1) & \longrightarrow & \mathrm{PSh}(\mathcal{A}_0), \end{array}$$

where π^* is given by precomposing with π [Lur17b, Proposition 5.5.8.10(3)]. The functor π^* admits a left adjoint $\pi_!$ [4] which, by the commutativity of the diagram of right adjoints above, is characterized by the commutativity of the corresponding diagram of left adjoints:

$$\begin{array}{ccc} \mathrm{PSh}(\mathcal{A}_0) & \xrightarrow{\pi_!} & \mathrm{PSh}(\mathcal{A}_1) \\ {\scriptstyle \mathrm{L}_{\Sigma}}\downarrow & & \downarrow{\scriptstyle \mathrm{L}_{\Sigma}} \\ \mathrm{PSh}_{\Sigma}(\mathcal{A}_0) & \longrightarrow & \mathrm{PSh}_{\Sigma}(\mathcal{A}_1). \end{array}$$

Using the universal property of PSh_{Σ} [Lur17b, Proposition 5.5.8.15], we also get that $\pi_!$ is characterized by the commutativity of

$$\begin{array}{ccc} \mathcal{A}_0 & \xrightarrow{\pi} & \mathcal{A}_1 \\ \downarrow & & \downarrow \\ \mathrm{PSh}_{\Sigma}(\mathcal{A}_0) & \xrightarrow{\pi_!} & \mathrm{PSh}_{\Sigma}(\mathcal{A}_1). \end{array}$$

These observations then apply to the additive functor

$$\overline{(-)} : \mathcal{A} \to \mathcal{A}/\mathcal{B},$$

which proves the assertion in (3).

To prove the fourth claim denote by $\mathrm{L}'(-)$ the functor given by the formula on the right. There is also a natural transformation $\mathrm{id}_{\mathrm{PSh}_{\Sigma}(\mathcal{A})} \xrightarrow{\alpha} \mathrm{L}'$. It suffices to prove that L' is a localization onto the subcategory of all $\mathrm{G} \in \mathrm{PSh}_{\Sigma}(\mathcal{A})$ such that $\mathrm{G}|_{\mathcal{B}} = 0$. Note that the two maps

$$\mathrm{F}(z) \otimes_{\mathbf{End}_{\mathcal{A}}(z)} \mathbf{Maps}_{\mathcal{A}}(z,z) \otimes_{\mathbf{End}_{\mathcal{A}}(z)} \mathbf{Maps}_{\mathcal{A}}(-,z) \longrightarrow \mathrm{F}(z) \otimes_{\mathbf{End}_{\mathcal{A}}(z)} \mathbf{Maps}_{\mathcal{A}}(-,z)$$

given by

$$f \otimes e \otimes e' \mapsto fe \otimes e' \text{ and } f \otimes e \otimes e' \mapsto f \otimes ee'$$

[4] Using, for example, the adjoint functor theorem [Lur17b, Theorem 5.5.2.9] where the accessibility hypothesis on π^* and the presentability hypotheses on the domain and target ∞-categories follows from [Lur17b, Proposition 5.5.8.10(1)].

are equivalences for any F, so by construction we have a commutative diagram in $\mathrm{Fun}^{\times}(\mathcal{C}^{\mathrm{op}}, \mathrm{Spt}_{\geqslant 0})$:

$$\begin{array}{ccccc}
\mathrm{F}(z) \otimes_{\mathbf{End}_{\mathcal{A}}(z)} \mathbf{Maps}_{\mathcal{A}}(z,z) \otimes_{\mathbf{End}_{\mathcal{A}}(z)} \mathbf{Maps}_{\mathcal{A}}(-,z) & \xrightarrow{\simeq} & \mathrm{F}(z) \otimes_{\mathbf{End}_{\mathcal{A}}(z)} \mathbf{Maps}_{\mathcal{A}}(-,z) & \longrightarrow & 0 \\
\downarrow \simeq & & \downarrow & & \downarrow \\
\mathrm{F}(z) \otimes_{\mathbf{End}_{\mathcal{A}}(z)} \mathbf{Maps}_{\mathcal{A}}(-,z) & \longrightarrow & \mathrm{F}(-) & \xrightarrow{\alpha_{\mathrm{F}}} & \mathrm{L}'\mathrm{F}(-) \\
\downarrow & & \downarrow \alpha_{\mathrm{F}} & & \downarrow \alpha_{\mathrm{L}'(\mathrm{F})} \\
0 & \longrightarrow & \mathrm{L}'\mathrm{F}(-) & \xrightarrow{\mathrm{L}'(\alpha)} & \mathrm{L}'\mathrm{L}'\mathrm{F}(-)
\end{array}$$

whose rows and columns are cofiber sequences. Hence $\mathrm{L}'(\alpha)(\mathrm{F})$ and $\alpha_{\mathrm{L}'(\mathrm{F})}$ are equivalences. Now by [Lur17b, Proposition 5.2.7.4] L' is a localization onto its image. The map

$$\mathrm{F}(z) \otimes_{\mathbf{End}_{\mathcal{A}}(z)} \mathbf{Maps}_{\mathcal{A}}(z,z) \longrightarrow \mathrm{F}(z)$$

is an equivalence, so $\mathrm{L}'\mathrm{F}(z) = 0$ and hence the image of L' is conatined in the image of L. Moreover, for any $\mathrm{G} \in \mathrm{PSh}_{\Sigma}(\mathcal{A})$ such that $\mathrm{G}|_{\mathcal{B}} = 0$

$$\mathrm{G}(z) \otimes_{\mathbf{End}_{\mathcal{A}}(z)} \mathbf{Maps}_{\mathcal{A}}(z,z) = 0,$$

so $\mathrm{G} \simeq \mathrm{L}'(\mathrm{G})$ is in the image of L'.

To prove the last claim it suffices to show that the functor

$$\mathrm{res}_{\mathrm{can}} \circ \mathrm{L} : \mathrm{PSh}_{\Sigma}(\mathcal{A}) \to \mathrm{PSh}_{\Sigma}(\mathcal{A}/\mathcal{B}) \to \mathrm{PSh}_{\Sigma}(\mathcal{A}),$$

is a colimit of functors

$$\mathrm{PSh}_{\Sigma}(\mathcal{A}) \xrightarrow{\mathrm{L}_{\mathrm{S}}} \mathrm{PSh}_{\Sigma}(\mathcal{A}/\langle z_1, \cdots, z_n\rangle) \xrightarrow{\mathrm{res}_{\mathrm{can},\mathrm{S}}} \mathrm{PSh}_{\Sigma}(\mathcal{A}),$$

over all finite subsets $\mathrm{S} = \{z_1, \cdots, z_n\} \subset \mathcal{B}$. Since for any $z \in \mathcal{B}$ there exists S such that $\mathrm{L}_{\mathrm{S}}\mathrm{F}(z) = 0$, the colimit lands in the image of L:

$$\mathrm{L}' : \mathrm{PSh}_{\Sigma}(\mathcal{A}) \to \mathrm{L}(\mathrm{PSh}_{\Sigma}(\mathcal{A})) \simeq \mathrm{PSh}_{\Sigma}(\mathcal{A}/\mathcal{B});$$

here the equivalence is due to part (2).

Now, since $\mathrm{res}_{\mathrm{can},\mathrm{S}}$ is right adjoint to L_{S}, the canonical map

$$\mathrm{Maps}(\mathrm{res}_{\mathrm{can},\mathrm{S}} \circ \mathrm{L}_{\mathrm{S}}\mathrm{F}, \mathrm{G}) \cong \mathrm{Maps}(\mathrm{F}, \mathrm{G})$$

is an equivalence for any G such that $\mathrm{G}(z) = 0$ for all $z \in \mathrm{S}$. In particular, if $\mathrm{G}|_{\mathcal{B}} = 0$, then we have the above equivalence for any S, so

$$\mathrm{Maps}(\mathrm{F}, \mathrm{G}) \cong \lim_{\mathrm{S}} \mathrm{Maps}(\mathrm{res}_{\mathrm{can},\mathrm{S}} \circ \mathrm{L}_{\mathrm{S}}\mathrm{F}, \mathrm{G}) \cong \mathrm{Maps}(\mathrm{L}'\mathrm{F}, \mathrm{G}),$$

since the limit is cofiltered and the diagram is constant. Now, note that for any such G we also have a canonical equivalence (by part (2) again):

$$\mathrm{Maps}(\mathrm{F}, \mathrm{G}) \cong \mathrm{Maps}(\mathrm{LF}, \mathrm{G}),$$

so by Yoneda lemma $\mathrm{LF} \cong \mathrm{L}'\mathrm{F}$. □

Corollary 2.2.4. *Let $\mathcal{A}$ be a small additive ∞-category and $\mathcal{B} \subset \mathcal{A}$ a full additive subcategory. For any $x, y \in \mathcal{A}$*

(1) the space $\mathrm{Maps}_{\mathcal{A}/\mathcal{B}}(x,y)$ can be computed as the Ω^{∞} of the connective spectrum

$$\mathrm{Cofib}(\operatorname*{colim}_{z_1,\cdots,z_n \in \mathcal{B}} \mathbf{Maps}_{\mathcal{A}}(\bigoplus_{i=1}^{n} z_i, y) \otimes_{\mathbf{End}(\bigoplus_{i=1}^{n} z_i)} \mathbf{Maps}_{\mathcal{A}}(x, \bigoplus_{i=1}^{n} z_i) \to \mathbf{Maps}_{\mathcal{A}}(x,y)).$$

(2) The abelian group $\pi_0 \mathrm{Maps}_{\mathcal{A}/\mathcal{B}}(x,y)$ can be computed as

$$\mathrm{Coker}(\bigoplus_{z \in \mathcal{B}} \pi_0 \mathrm{Maps}_{\mathcal{A}}(z,y) \otimes \pi_0 \mathrm{Maps}_{\mathcal{A}}(x,z) \to \pi_0 \mathrm{Maps}_{\mathcal{A}}(x,y)).$$

Proof. The first claim follows directly from Theorem 2.2.3(3 and 5) and taking Ω^∞ to obtain the mapping space from the mapping spectrum. This claim in particular implies that $\pi_0 \operatorname{Maps}_{\mathcal{A}/\mathcal{B}}(x, y)$ can be computed as the quotient of $\pi_0 \operatorname{Maps}_{\mathcal{A}}(x, y)$ modulo the subgroup of those morphisms that factor through $z \in \mathcal{B}$. This is exactly what formula (2) says. □

Remark 2.2.5. A very similar formula to the one in Corollary 2.2.4(1) was obtained in the setting of dg-quotients of dg-categories in [Dri02].

2.3. **Idempotents.** We now discuss the condition of idempotent completeness in the context of additive ∞-categories. Recall that we have the free category containing an idempotent, Idem, which is a subcategory of Idem^+, the free category containing a retraction; these categories are constructed in [Lur17b, Definition 4.4.5.2]. An **idempotent** in an ∞-category $\mathcal{C}$ is a functor $e : \mathrm{Idem} \to \mathcal{C}$ and we say that e is **effective** if e extends to a functor $\mathrm{Idem}^+ \to \mathcal{C}$. We say that $\mathcal{C}$ is **idempotent complete** if every idempotent is effective. Following [Lur17b, Corollary 4.4.5.14] the functor $e : \mathrm{Idem} \to \mathcal{C}$ admits a colimit if and only if e is an effective idempotent.

Remark 2.3.1. An idempotent e in $\mathcal{C}$ gives rise to an idempotent of $h(\mathcal{C})$ which can be identified with a morphism e in $h(\mathcal{C})$ such that $e \circ e = e$. In general, this datum is not sufficient for defining an idempotent [Lur17b, Counterexample 4.4.5.19]. However, if $\mathcal{C}$ is a stable ∞-category, then $\mathcal{C}$ is idempotent complete if and only if $h(\mathcal{C})$ is [Lur17a, Lemma 1.2.4.6] and, in fact, any idempotent in $h(\mathcal{C})$ lifts to one in $\mathcal{C}$ [Lur17a, Warning 1.2.4.8]. We observe that the same proof also works for additive ∞-categories.

Proposition 2.3.2. *For an additive ∞-category $\mathcal{A}$ any idempotent in $h(\mathcal{A})$ can be lifted to an idempotent in $\mathcal{A}$. Consequently, the following are equivalent:*

(1) $\mathcal{A}$ is idempotent-complete

(2) $h(\mathcal{A})$ is idempotent-complete

(3) any morphism $x \xrightarrow{e} x \in \mathcal{A}$ such that there is a homotopy $e \circ e \simeq e$ is equivalent to a morphism of the form

$$x_1 \oplus x_2 \xrightarrow{\begin{pmatrix} \mathrm{id}_{x_1} & 0 \\ 0 & 0 \end{pmatrix}} x_1 \oplus x_2$$

Proof. Let

$$e : x \to x,$$

be an idempotent in $h(\mathcal{A})$, i.e., a morphism e equipped with a homotopy $h : e \to e \circ e$. Since $\mathcal{A}$ is an additive ∞-category, there is a fully faithful Yoneda functor $\mathcal{A} \to \widehat{\mathcal{A}}$ by Corollary 2.2.2. We consider the morphism e in $\mathrm{Fun}^\times(\mathcal{A}^{\mathrm{op}}, \mathrm{Spt})$, which defines an idempotent in $h(\widehat{\mathcal{A}})$.

To prove the result we will promote e to a weak retraction diagram in the sense of [Lur17b, Definition 4.4.5.4(2), Remark 4.4.5.5] which we briefly review. Let $\mathrm{Ret} \subset \mathrm{Idem}^+$ be the simplicial set classifying retractions (as in [Lur17b, Page 304]). Then a weak retraction diagram is a functor out of Ret. The inclusion $\mathrm{Ret} \subset \mathrm{Idem}^+$ is an inner anodyne map [Lur17b, Proposition 4.4.5.6] and thus for any ∞-category $\mathcal{C}$ the induced functor $\mathrm{Fun}(\mathrm{Idem}^+, \mathcal{C}) \to \mathrm{Fun}(\mathrm{Ret}, \mathcal{C})$ is trivial fibration of simplicial sets [Lur17b, Corollary 4.4.5.7]. Concretely, any weak retraction diagram extends, in an essentially unique way, to a functor out of Idem^+. We first claim that the idempotent e in $h(\widehat{\mathcal{A}})$ extends to a map of simplicial sets $\mathrm{Ret} \to \widehat{\mathcal{A}}$ and thus to a functor out of Idem^+.

The argument is the same as in the second paragraph of the proof of [Lur17a, Lemma 1.2.4.6] which we sketch for the reader's convenience. Denote by x_0 (resp. x_1) the sequential colimit

$$x \xrightarrow{e} x \xrightarrow{e} x \to \cdots$$

$$(\text{resp. } x \overset{1-e}{\to} x \overset{1-e}{\to} x \to \cdots),$$

in $\widehat{\mathcal{A}}$[5]. Since the latter ∞-category is cocomplete, these colimits exist. Now, for any $\mathrm{F} \in \widehat{\mathcal{A}}$, we note that e induces an idempotent on the abelian group $\pi_i\mathrm{F}(x)$ for all $i \in \mathbb{Z}$ so that we have a splitting:

$$\pi_i\mathrm{F}(x) \simeq \pi_i\mathrm{F}(x)_+ \oplus \pi_i\mathrm{F}(x)_-,$$

where e acts as the identity on $\pi_i\mathrm{F}(x)_+$ and as 0 on $\pi_i\mathrm{F}(x)_-$. Thus, the tower of abelian groups

$$\cdots \pi_i\mathrm{F}(x) \xrightarrow{e^*} \pi_i\mathrm{F}(x) \xrightarrow{e^*} \pi_i\mathrm{F}(x),$$

splits as a direct sum of towers:

$$\cdots \pi_i\mathrm{F}(x)_+ \xrightarrow{\mathrm{id}} \pi_i\mathrm{F}(x)_+ \xrightarrow{\mathrm{id}} \pi_i\mathrm{F}(x)_+,$$

and

$$\cdots \pi_i\mathrm{F}(x)_- \overset{0}{\to} \pi_i\mathrm{F}(x)_- \overset{0}{\to} \pi_i\mathrm{F}(x)_-.$$

such that

$$\lim \pi_i\mathrm{F}(x) \simeq \pi_i\mathrm{F}(x)_+,$$

and the $\lim^1$-term vanishes. This means that the canonical map $x \to x_0$ induces an isomorphism of abelian groups, for all $i \in \mathbb{Z}$:

$$\pi_i\mathrm{F}(x_0) \simeq \pi_i(\mathrm{maps}(\mathrm{colim}\, x, \mathrm{F})) \simeq \lim \pi_i\mathrm{F}(x) \simeq \pi_i\mathrm{F}(x)_+.$$

By a similar argument for the idempotent $1-e$, we have

$$\pi_i\mathrm{F}(x_1) \simeq \pi_i\mathrm{F}(x)_-.$$

Therefore the canonical map $x \to x_0 \oplus x_1$ defines an isomorphism for all F and all $i \in \mathbb{Z}$:

$$\pi_i\mathrm{F}(x) \cong \pi_i\mathrm{F}(x_0) \oplus \pi_i\mathrm{F}(x_1),$$

and thus induces a splitting:

$$\mathrm{F}(x) \simeq \mathrm{F}(x_0) \oplus \mathrm{F}(x_1).$$

Since this is true for all F, we also have a splitting of representable presheaves:

$$x \simeq x_0 \oplus x_1.$$

Furthermore, we see that the inclusion of summand, $x_0 \to x$, together with the canonical map to the colimit, $x \to x_0$, induces a weak retraction diagram

$$\begin{array}{ccccc} & & x & & \\ & \nearrow & & \searrow & \\ x_0 & & \xrightarrow{\quad\mathrm{id}\quad} & & x_0, \end{array}$$

in $\widehat{\mathcal{A}}$ as claimed. To see that the idempotent e in $h(\mathcal{A})$ lifts to one on $\mathcal{A}$, observe that the corresponding functor $\mathrm{Idem}^+ \to \widehat{\mathcal{A}}$ restricts to a functor landing in the Yoneda image:

$$\mathrm{Idem} \to \mathcal{A} \subset \widehat{\mathcal{A}};$$

note that we have used the additivity of $\mathcal{A}$ to get the fully faithfulness of the Yoneda functor; see Corollary 2.2.2.

We now prove that the three assumptions are equivalent. The third assumption is clearly equivalent to the second assumption. Since any idempotent in $h(\mathcal{A})$ lifts to an idempotent in $\mathcal{A}$, it must be effective if the lift is effective in $\mathcal{A}$, so we have (1) $\Rightarrow$ (2). Now we prove (2) $\Rightarrow$ (1). Let $e\colon \mathrm{Idem} \to \mathcal{A}$ be an idempotent in $\mathcal{A}$. Denote by x the colimit of e in $\widehat{\mathcal{A}}$. It is a retract of an object of $\mathcal{A}$, so by the assumption its image in $h(\widehat{\mathcal{A}})$ is isomorphic to an object of $h(\mathcal{A})$ considered as a full subcategory of $h(\widehat{\mathcal{A}})$. So the colimit of e is equivalent to an object of $\mathcal{A}$ and e is effective in $\mathcal{A}$. □

[5]We note that the existence of the map $1-e$ is where the additivity of $\widehat{\mathcal{A}}$ gets used.

Proposition 2.3.2 justifies the usage of some standard terminology from the theory of additive categories in the context of additive ∞-categories.

Definition 2.3.3. Let $\mathcal{A}$ be an additive ∞-category.

(1) We say that an idempotent $e : x \to x$ in $\mathcal{A}$ **splits** if it is homotopic to a morphism of the form
$$x_1 \oplus x_2 \xrightarrow{\begin{pmatrix} \mathrm{id}_{x_1} & 0 \\ 0 & 0 \end{pmatrix}} x_1 \oplus x_2.$$

(2) A full subcategory $\mathcal{A} \subset \mathcal{B}$ is said to be **retract-closed** if every idempotent of an object of $\mathcal{A}$ that splits in $\mathcal{B}$ also splits in $\mathcal{A}$.

(3) If $\mathcal{A}$ is an additive ∞-category, then we say that $\mathcal{A}$ is **idempotent complete** if each idempotent $e : x \to x$ in $\mathcal{A}$ splits.

Lemma 2.3.4. *Let $\mathcal{A}$ be a small additive ∞-category. Then there exists a small additive ∞-category $\mathrm{Kar}(\mathcal{A})$ and an additive functor $\mathcal{A} \to \mathrm{Kar}(\mathcal{A})$ which is an initial functor among all idempotent complete additive ∞-categories receiving an additive functor from $\mathcal{A}$. We call this category the idempotent completion of $\mathcal{A}$.*

Proof. According to [Lur17b, Proposition 5.1.4.2] and the identifications of Proposition 2.3.2, we can take the embedding
$$\mathcal{A} \to \mathrm{PSh}_\Sigma(\mathcal{A}),$$
and set $\mathrm{Kar}(\mathcal{A})$ to be the full subcategory of $\mathrm{PSh}_\Sigma(\mathcal{A})$ spanned by functors which are retracts of the Yoneda image. Note that this is also an additive ∞-category. □

Definition 2.3.5. If $f : \mathcal{A} \to \mathcal{B}$ is an additive functor between small additive ∞-categories, then we get a functor $\mathrm{Kar}(f) : \mathrm{Kar}(\mathcal{A}) \to \mathrm{Kar}(\mathcal{B})$. We say that that f is an **equivalence up to idempotent completion** if $\mathrm{Kar}(f)$ is an equivalence. We will also denote the subcategory of additive ∞-categories spanned by those which are idempotent complete as:
$$\mathbf{Cat}_\infty^{\mathrm{Kar}} \subset \mathbf{Cat}_\infty^{\mathrm{add}}.$$

Definition 2.3.6. We say that an additive ∞-category $\mathcal{A}$ is weakly idempotent complete if for any $\mathrm{X}, \mathrm{Y} \in \mathcal{A}$ such that
$$\mathrm{X} \cong \mathrm{Y} \oplus \mathrm{V}$$
in $\mathrm{Kar}(\mathcal{A})$ for some $\mathrm{V} \in \mathrm{Kar}(\mathcal{A})$, there exists an object of $\mathcal{A}$ equivalent to V.

We denote the subcategory of additive ∞-categories spanned by those which are weakly idempotent complete as:
$$\mathbf{Cat}_\infty^{\mathrm{wKar}} \subset \mathbf{Cat}_\infty^{\mathrm{add}}.$$

2.4. **Exact sequences.**

Definition 2.4.1. Let $\mathcal{A}, \mathcal{C}$ be small additive ∞-categories and $\mathcal{B} \subset \mathcal{A}$ be an additive full subcategory. We say that the sequence
$$\mathcal{B} \xrightarrow{i} \mathcal{A} \xrightarrow{j} \mathcal{C}$$
is an **exact sequence of additive ∞-categories** if (1) the composite is null and (2) the induced map $\mathcal{A}/\mathcal{B} \to \mathcal{C}$ is an equivalence up to idempotent completion.

Note that in case the composite is null, the space of choices of an isomorphism $j \circ i \simeq 0$ is contractible, so the condition (1) is actually a property and does not require any additional choices.

2.4.2. *Comparison with exact sequences in stable ∞-categories.* We now explain the passage of going from small additive ∞-categories to small stable ∞-categories. We denote by $\mathbf{Cat}_\infty^{\mathrm{st}}$ the ∞-category of small stable ∞-categories and exact functors between them. From an additive ∞-category we can extract an object of $\mathbf{Cat}_\infty^{\mathrm{st}}$ by the next procedure.

Definition 2.4.3. Let $\mathcal{A}$ be a small additive ∞-category. A small stable ∞-category $\mathcal{A}^{\mathrm{fin}}$ together with a functor $\mathcal{A} \to \mathcal{A}^{\mathrm{fin}}$ satisfying the universal property

$$\mathrm{Fun}^{\times}(\mathcal{A}, \mathcal{C}) \cong \mathrm{Fun}^{\mathrm{ex}}(\mathcal{A}^{\mathrm{fin}}, \mathcal{C})$$

for any stable ∞-category $\mathcal{C}$, is called the ∞-category of **finite cell $\mathcal{A}$-modules**. If $\mathcal{A}^{\mathrm{fin}}$ exists, we should regard $\mathcal{A}^{\mathrm{fin}}$ as the "free stable ∞-category" generated by $\mathcal{A}$.

Remark 2.4.4. The ∞-category of finite cell $\mathcal{A}$-modules always exists: consider the Spanier-Whitehead stabilization of the ∞-category $\mathrm{PSh}_\Sigma^{\mathrm{fin}}(\mathcal{A})$ — the smallest subcategory of $\mathrm{PSh}_\Sigma(\mathcal{A})$ containing all representable presheaves and closed under finite colimits. Indeed, combining [Lur17b, Proposition 5.3.6.2] and [Lur18, Proposition C.1.1.7] we get the required universal property for $\mathcal{A}^{\mathrm{fin}}$:

$$\mathrm{Fun}^{\times}(\mathcal{A}, \mathcal{C}) \cong \mathrm{Fun}^{\mathrm{ex}}(\mathcal{A}^{\mathrm{fin}}, \mathcal{C})$$

for any stable ∞-category $\mathcal{C}$. In particular, we obtain a functor

$$(-)^{\mathrm{fin}} : \mathbf{Cat}_\infty^{\mathrm{add}} \to \mathbf{Cat}_\infty^{\mathrm{st}}, \tag{2.4.5}$$

that is left adjoint to the forgetful functor.

Alternatively, $\mathcal{A}^{\mathrm{fin}}$ can be described as a full subcategory of $\widehat{\mathcal{A}}$ (this follows from [Lur18, Remark C.1.1.6]) and the image coincides with the smallest subcategory of $\widehat{\mathcal{A}}$ containing representable presheaves and closed under taking shifts and finite colimits.

Example 2.4.6. Let R be a connective $\mathbb{E}_1$-ring spectrum. In the ∞-category of connective R-module spectra $\mathbf{RMod}_{\mathrm{R}}^{\mathrm{cn}}$, we have the subcategory of **projective** R**-modules**

$$\mathbf{Proj}_{\mathrm{R}} \subset \mathbf{RMod}_{\mathrm{R}}^{\mathrm{cn}},$$

defined as the subcategory of projective objects in the sense of [Lur17b, Definition 5.5.8.18]. These objects can also be characterized as retracts of free R-modules [Lur17a, Proposition 7.2.2.7]. Inside $\mathbf{Proj}_{\mathrm{R}}$ there is the subcategory of finite generated ones

$$\mathbf{Proj}_{\mathrm{R}}^{\mathrm{fg}} \subset \mathbf{Proj}_{\mathrm{R}},$$

which can be characterized as those with π_0 being finitely generated as a π_0R-module [Lur17a, Corollary 7.2.2.9]. The ∞-category $\mathbf{Proj}_{\mathrm{R}}^{\mathrm{fg}}$ is small and idempotent complete.

Let us now recall the theory of exact sequences in small stable ∞-categories. We denote, following our notation for additive ∞-categories, by $\mathrm{Kar}(\mathcal{C})$ the idempotent completion of a stable ∞-category $\mathcal{C}$ (see [BGT13, Section 2.2] for a treatment in this context where they write $\mathrm{Idem}(\mathcal{C})$ for $\mathrm{Kar}(\mathcal{C})$). Suppose that we have a sequence of functors

$$\mathcal{B} \to \mathcal{A} \to \mathcal{C}$$

in $\mathbf{Cat}_\infty^{\mathrm{st}}$. Then we say it is **exact** if the composite is zero, $\mathcal{B} \to \mathcal{A}$ is fully faithful and the functor

$$\mathcal{A}/\mathcal{B} \to \mathcal{C}$$

becomes an equivalence after idempotent completion.

Lemma 2.4.7. *The functor (2.4.5) sends exact sequences of additive ∞-categories to exact sequences of stable ∞-categories.*

Proof. Suppose that we have an exact sequence of additive ∞-categories

$$\mathcal{B} \to \mathcal{A} \to \mathcal{C}.$$

We consider the diagram in $\mathbf{Cat}^{\mathrm{st}}_\infty$:

$$\mathcal{B}^{\mathrm{fin}} \to \mathcal{A}^{\mathrm{fin}} \to \mathcal{C}^{\mathrm{fin}}.$$

Fully faithfulness of $\mathcal{B}^{\mathrm{fin}} \to \mathcal{A}^{\mathrm{fin}}$ follows, for example, from [Lur17b, Proposition 5.3.5.11]. It follows from Theorem 2.2.3(2) that $\mathcal{A} \to \mathcal{A}/\mathcal{B}$ can be identified with the cofiber of the map $\mathcal{B} \to \mathcal{A}$ in $\mathbf{Cat}^{\mathrm{add}}_\infty$. As discussed in Remark 2.4.4 $(-)^{\mathrm{fin}}$ is a left adjoint, so it preserves cofibers. Lastly, note that

$$\mathrm{Kar}((-)^{\mathrm{fin}}) \simeq (\mathrm{Kar}(-))^{\mathrm{fin}},$$

since both functors are left adjoint to the forgetful functor $\mathbf{Cat}^{\mathrm{perf}}_\infty \to \mathbf{Cat}^{\mathrm{add}}_\infty$.

Therefore, we conclude that the functor

$$\mathrm{Kar}(\mathcal{A}^{\mathrm{fin}}/\mathcal{B}^{\mathrm{fin}}) \simeq \mathrm{Kar}((\mathcal{A}/\mathcal{B})^{\mathrm{fin}}) \to \mathrm{Kar}(\mathcal{C}^{\mathrm{fin}})$$

is indeed an equivalence. □

3. Weight structures

This section is devoted to weight structures which is the central notion of this thesis. In a precise sense it is a bridge between the world additive ∞-categories and stable ∞-categories.

3.1. **Basics.**

Our goal in this section is to explain how one can recognize objects in the essential image of the functor (2.4.5). This is done via Bondarko's theory of weight structures, which we have already introduced in Section 1 (see Definition 1.1.1).

Definition 3.1.1. We define

$$\mathbf{WCat}^{\mathrm{st},b}_\infty$$

to be the full subcategory of the pullback ∞-category:

$$\mathrm{Maps}(\{\bullet_{\leqslant 0} \to \bullet \leftarrow \bullet_{\geqslant 0}\}, \mathbf{Cat}_\infty) \times_{\mathbf{Cat}_\infty} \mathbf{Cat}^{\mathrm{st}}_\infty,$$

spanned by the triples $(\mathrm{F}, \mathrm{F}(\bullet), \mathrm{F}(\bullet) \xrightarrow{\mathrm{id}} \mathrm{F}(\bullet))$ such that the morphisms $\mathrm{F}(\bullet_{\leqslant 0}) \to \mathrm{F}(\bullet)$ and $\mathrm{F}(\bullet_{\geqslant 0}) \to \mathrm{F}(\bullet)$ are embeddings of full subcategories satisfying the axioms of a bounded weight structure (see also Definition 1.1.1):

(1) $\Sigma\mathrm{F}(\bullet_{\geqslant 0}) \subset \mathrm{F}(\bullet_{\geqslant 0})$, $\Omega\mathrm{F}(\bullet_{\leqslant 0}) \subset \mathrm{F}(\bullet_{\leqslant 0})$
(2) if $x \in \mathrm{F}(\bullet_{\leqslant 0}), y \in \Sigma\mathrm{F}(\bullet_{\geqslant 0})$ then
$$\pi_0 \mathrm{Maps}_{\mathcal{C}}(x, y) \simeq 0,$$
(3) for any object $x \in \mathcal{C}$ we have a cofiber sequence
$$x_{\leqslant 0} \to x \to x_{\geqslant 1},$$
where $x_{\leqslant 0} \in \mathrm{F}(\bullet_{\leqslant 0})$ and $x_{\geqslant 1} \in \Sigma\mathrm{F}(\bullet_{\geqslant 0})$,
(4) $\mathrm{F}(\bullet) = \bigcup_n \Omega^{-n}\mathrm{F}(\bullet_{\geqslant 0}) \cap \Sigma^n\mathrm{F}(\bullet_{\leqslant 0})$.

In other words this is the ∞-category of small, boundedly weighted stable ∞-categories and weight exact functors. Note that since the functors $\mathrm{F}(\bullet_{\leqslant 0}) \to \mathrm{F}(\bullet)$ and $\mathrm{F}(\bullet_{\geqslant 0}) \to \mathrm{F}(\bullet)$ are fully faithful, the forgetful functor

$$\mathbf{WCat}^{\mathrm{st},b}_\infty \to \mathbf{Cat}^{\mathrm{st}}_\infty$$

induces fully faithful functors on mapping spaces.

Remark 3.1.2. Besides being retract-closed, $\mathcal{C}_{w\geqslant 0}$ and $\mathcal{C}_{w\leqslant 0}$ are also extension stable: if $x \to y \to z$ is a cofiber sequence in $\mathcal{C}$ where $x, z \in \mathcal{C}_{w\geqslant 0}$ then so is y. The same goes for $\mathcal{C}_{w\leqslant 0}$; we refer to [Bon10, Proposition 1.3.3(3)] for a proof.

Remark 3.1.3. Bounded weight structures are quite easy to construct. Let $\mathcal{C}$ be a stable ∞-category. Assume we are given with a subcategory $\mathrm{N} \subset \mathcal{C}$ such that

(1) N generates $\mathcal{C}$ under finite limits, finite colimits and retracts,

(2) N is **negatively self-orthogonal**, that is, for any $x, y \in \mathrm{N}$ the mapping spectrum

$$\mathrm{maps}_{\mathcal{C}}(x, y),$$

is connective. Equivalently,

$$\pi_0 \operatorname{Maps}_{\mathcal{C}}(x, \Sigma^n y) = 0,$$

for $n > 0$.

According to [BS18a, Corollary 2.1.2], defining

$$\mathcal{C}_{w\geqslant 0} = \{\text{retracts of finite colimits of objects of N}\}$$

$$\mathcal{C}_{w\leqslant 0} = \{\text{retracts of finite limits of objects of N}\}$$

gives a weight structure on $\mathcal{C}$ whose heart is the minimal retract-closed additive subcategory containing N. Note that it is a bounded weight structure. Indeed, the subcategory

$$\bigcup_n (\mathcal{C}_{\geqslant -n} \cap \mathcal{C}_{\leqslant n}) \subset \mathcal{C}$$

contains N and is closed under finite limits, finite colimits and retracts, so by assumption (1) the inclusion is an equality. In fact it is a unique weight structure whose heart contains N as a subcategory.

Conversely, the heart of a bounded weight structure on $\mathcal{C}$ trivially satisfies (2) and it follows by induction using weight decompositions that $\mathcal{C}$ is generated by the objects of the heart (see [Bon10, Corollary 1.5.7]).

Example 3.1.4. Let R be a connective $\mathbb{E}_1$-ring spectrum. Then the construction of Remark 3.1.2 with

$$\mathrm{N} = \{\mathrm{R}\} \subset \mathbf{Perf}_{\mathrm{R}}$$

gives a weight structure on the category $\mathbf{Perf}_{\mathrm{R}}$. In this case

$$\mathrm{M} \in (\mathbf{Perf}_{\mathrm{R}})_{w\geqslant 0} \iff \mathrm{M} \text{ is connective},$$

$$\mathrm{M} \in (\mathbf{Perf}_{\mathrm{R}})_{w\leqslant 0} \iff \pi_0 \operatorname{Maps}_{\mathrm{R}}(\mathrm{M}, \Sigma \mathrm{N}) = 0 \text{ for any connective N}.$$

the subcategory of connective modules and the subcategory of modules with trivial positive R-cohomology. Compatibly with what Warning 3.1.5(1) suggested in the case $\mathrm{R} = \mathbb{S}$, the weight decompositions correspond precisely to cellular filtrations.

Warning 3.1.5. *The notion of a weight structure is visibly similar to the notion of a t-structure, and in many aspects their basic properties are also similar, however, one must be aware of some important differences:*

(1) Unlike t-truncations, weight decompositions are not functorial. In Example 3.1.4 for $\mathrm{R} = \mathbb{S}$ *weight truncations correspond precisely to cellular filtrations. In particular, they depend on the choice of a cellular structure. In Example 1.1.5 weight truncations correspond to brutal truncations of a complex. One might think of a choice of weight decompositions as of a choice of a projective resolution of a given object.*

(2) *As will be explained in the next section, the heart of a boundedly weighted stable ∞-category, as an additive ∞-category, determines the stable ∞-category up to equivalence. This contrasts the case of bounded t-structures where many non-equivalent stable ∞-categories with a t-structure might have equivalent hearts as abelian categories.*

3.2. **From additive ∞-categories to stable ∞-categories via weights.** In order to use weight structures to detect the image of $(-)^{\mathrm{fin}}$, we factor it through $\mathbf{WCat}_\infty^{\mathrm{st},b}$.

Construction 3.2.1. We construct a factorization:

$$\begin{array}{ccc} & & \mathbf{WCat}_\infty^{\mathrm{st},b} \\ & \overset{((-)^{\mathrm{fin}},w)}{\nearrow} & \downarrow \\ \mathbf{Cat}_\infty^{\mathrm{add}} & \xrightarrow[(-)^{\mathrm{fin}}]{} & \mathbf{Cat}_\infty^{\mathrm{st}}. \end{array} \tag{3.2.2}$$

Here, the right vertical map is the forgetful functor. To construct the dashed arrow, it suffices to equip $\mathcal{A}^{\mathrm{fin}}$ with a canonical bounded weight structure (note that the vertical functor induces fully faithful functors on the mapping spaces). To do so, we use Remark 3.1.3. Let N be the essential image of $\mathcal{A} \to \mathcal{A}^{\mathrm{fin}}$. By definition, N generates $\mathcal{A}^{\mathrm{fin}}$ under finite limits, finite colimits and retracts. To check the self-orthogonality condition: for $n > 0$ we have for any x, y in $\mathcal{A}$ (which we abusively identify with the essential image of $\mathcal{A} \to \mathcal{A}^{\mathrm{fin}}$):

$$\pi_0(\mathrm{maps}_{\mathcal{A}^{\mathrm{fin}}}(x, \Sigma^n y)) \simeq \pi_0(\Sigma^n \mathrm{maps}_{\mathcal{A}^{\mathrm{fin}}}(x, y)) \simeq \pi_{-n}(\mathbf{Maps}_{\mathcal{A}}(x, y)) \simeq 0.$$

Here we use the convention of Theorem 2.2.3 where $\mathbf{Maps}(-, y)$ indicate a functor landing in connective spectra; this also explains the last equivalence.

Theorem 3.2.3. *We have an adjoint pair*

$$((-)^{\mathrm{fin}}, w) : \mathbf{Cat}_\infty^{\mathrm{add}} \rightleftarrows \mathbf{WCat}_\infty^{\mathrm{st},b} : (-)^{\heartsuit_w}. \tag{3.2.4}$$

Proof. By Remark 2.4.4 we have a binatural equivalence

$$\mathrm{Fun}^{\times}(\mathcal{A}, \mathcal{C}) \simeq \mathrm{Fun}^{\mathrm{ex}}(\mathcal{A}^{\mathrm{fin}}, \mathcal{C}) \tag{3.2.5}$$

for any additive ∞-category $\mathcal{A}$ and stable ∞-category $\mathcal{C}$. Now suppose $\mathcal{C}$ is equipped with a bounded weight structure u. Then,

(1) we have a subspace inclusion

$$\mathrm{Maps}_{\mathbf{Cat}_\infty^{\mathrm{add}}}(\mathcal{A}, \mathcal{C}^{\heartsuit_u}) \subset \mathrm{Fun}^{\times}(\mathcal{A}, \mathcal{C}),$$

given by additive functors $\mathcal{A} \to \mathcal{C}$ which land in $\mathcal{C}^{\heartsuit_u}$;

(2) we have a subspace inclusion

$$\mathrm{Maps}_{\mathbf{WCat}_\infty^{\mathrm{st},b}}((\mathcal{A}^{\mathrm{fin}}, w), (\mathcal{C}, u)) \subset \mathrm{Fun}^{\mathrm{ex}}(\mathcal{A}^{\mathrm{fin}}, \mathcal{C}),$$

given by those exact functors which are furthermore weight exact.

Under the equivalence of (3.2.5), these subspaces are identified:

$$\mathrm{Maps}_{\mathbf{Cat}_\infty^{\mathrm{add}}}(\mathcal{A}, \mathcal{C}^{\heartsuit_u}) \simeq \mathrm{Maps}_{\mathbf{WCat}_\infty^{\mathrm{st},b}}((\mathcal{A}^{\mathrm{fin}}, w), (\mathcal{C}, u)).$$

which gives us the desired adjunction. □

Proposition 3.2.6. *Let $(\mathcal{C}, w_{\mathcal{C}}), (\mathcal{D}, w_{\mathcal{D}})$ be boundedly weighted stable ∞-categories. The counit of the adjunction (3.2.4) gives an equivalence in $\mathbf{WCat}_\infty^{\mathrm{st},b}$*

$$((\mathcal{C}^{\heartsuit_w})^{\mathrm{fin}}, w) \simeq (\mathcal{C}, w_{\mathcal{C}}).$$

Consequently $(-)^{\heartsuit_w}$ is fully faithful.

Proof. Consider the functor

$$\mathrm{F} : \mathcal{C} \to \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Spt}) \to \mathrm{Fun}^{\times}((\mathcal{C}^{\heartsuit_w})^{\mathrm{op}}, \mathrm{Spt}).$$

By the orthogonality axiom for weight structures for any $\mathrm{X}, \mathrm{Y} \in \mathcal{C}^{\heartsuit_{w_{\mathcal{C}}}}$ the map

$$\mathbf{Maps}_{\mathcal{C}^{\heartsuit_w}}(\mathrm{Y}, \mathrm{X}) \simeq \tau_{\geqslant 0}\mathbf{maps}_{\mathcal{C}}(\mathrm{Y}, \mathrm{X}) \to \mathbf{maps}_{\mathcal{C}}(\mathrm{Y}, \mathrm{X}) = \mathrm{F}(\mathrm{X})(\mathrm{Y})$$

is an equivalence of spectra (see notation after Corollary 2.2.2). In particular, the composite

$$\mathcal{C}^{\heartsuit_w} \to \mathcal{C} \to \widehat{(\mathcal{C}^{\heartsuit_w})}$$

is equivalent to the spectral Yoneda embedding (see Corollary 2.2.2). Moreover, F is exact, so by the universal property (see Remark 2.4.4) the composite functor

$$(\mathcal{C}^{\heartsuit_w})^{\mathrm{fin}} \to \mathcal{C} \to \widehat{(\mathcal{C}^{\heartsuit_w})}$$

is equivalent to the canonical embedding $(\mathcal{C}^{\heartsuit_w})^{\mathrm{fin}} \subset \widehat{(\mathcal{C}^{\heartsuit_w})}$. Since F is fully faithful by construction, we obtain that the counit functor is fully faithful.

Since w is bounded $\mathcal{C}$ is generated by $\mathcal{C}_w^{\heartsuit}$ under finite colimits and shifts, so the image of F is contained in $(\mathcal{C}^{\heartsuit_w})^{\mathrm{fin}}$. This ensures that both F and the counit functor are essentially surjective. □

We now prove the main theorem of this section.

Theorem 3.2.7. *The adjoint pair (3.2.4) restricts to an equivalence of ∞-categories*

$$\mathbf{Cat}_{\infty}^{\mathrm{wKar}} \rightleftarrows \mathbf{WCat}_{\infty}^{\mathrm{st},b}.$$

Proof. After Proposition 3.2.6 it suffices to show that the essential image of $(-)^{\heartsuit_w}$ consists of those additive ∞-categories that are weakly idempotent complete. All additive ∞-categories in the image are weakly idempotent complete by Theorem 4.3.2(II.2) of [Bon10].

Conversely, we show that for a weakly idempotent complete additive category $\mathcal{A}$ the heart of the weight structure on $\mathcal{A}^{\mathrm{fin}}$ is $\mathcal{A}$. By Remark 3.1.3 it equals to the retract-closure of $\mathcal{A}$ in $\mathcal{A}^{\mathrm{fin}}$ which is equal to $\mathcal{A}$ by Theorem 4.3.2(II.2) of [Bon10].

□

Corollary 3.2.8. *For an additive category $\mathcal{A}$ there is an equivalence of (weighted) stable ∞-categories*

$$\mathcal{A}^{\mathrm{fin}} \simeq \mathrm{Com}^b(\mathcal{A}).$$

In other words, $\mathrm{Com}^b(\mathcal{A})$ *is the free stable ∞-category on an additive category.*

Proof. As Example 1.1.5 shows $\mathrm{Com}^b(\mathcal{A})$ admits a weight structure, whose heart is $\mathcal{A}$. Now the result follows directly from Proposition 3.2.6. □

Corollary 3.2.9. *For a connective $\mathbb{E}_1$-ring spectrum* R *we have a natural equivalence*

$$(\mathbf{Proj}_{\mathrm{R}}^{\mathrm{fg}})^{\mathrm{fin}} \simeq \mathbf{Perf}_{\mathrm{R}}.$$

Proof. Similarly, the result follows directly from applying Proposition 3.2.6 to $\mathbf{Perf}_{\mathrm{R}}$ endowed with the weight structure from Example 3.1.4. □

The next corollary shows that our result is compatible wih the theory of weight structures on triangulated categories, giving a conceptual explanation to a very useful ad-hoc construction introduced by Bondarko. The reader unfamiliar with this theory may feel free to skip this part.

Corollary 3.2.10. *For any boundedly weighted stable ∞-category $\mathcal{C}$ there is a natural exact functor $\mathcal{C} \xrightarrow{t} \mathrm{Com}^b(h\mathcal{C}^{\heartsuit})$ that induces the weight complex functor in the sense of* [Bon10, Section 3]

$$t^{\mathrm{B}} : h(\mathcal{C}) \to \mathrm{K}^b(h\mathcal{C}^{\heartsuit}) \to \mathrm{K}^b_w(h\mathcal{C}^{\heartsuit}).$$

Proof. For an additive ∞-category $\mathcal{A}$ we have a natural functor

$$\mathcal{A} \to h\mathcal{A}.$$

In view of Theorem 3.2.7 this induces a natural functor

$$\mathcal{C} \simeq (\mathcal{C}^{\heartsuit})^{\mathrm{fin}} \to (h\mathcal{C}^{\heartsuit})^{\mathrm{fin}} \simeq \mathrm{Com}^b(h\mathcal{C}^{\heartsuit})$$

where the equivalence on the right is from Corollary 3.2.8. We define t to be the unit of the adjunction (3.2.4). Bondarko's functor t^{B} was defined via choosing weight decompositions for all objects and taking the fibers between neighboring stages to obtain the components of a complex. It was then shown that the construction does not depend on the choices and is functorial. In particular, the weight complex functor for the triangulated category $\mathrm{K}^b(h\mathcal{C}^{\heartsuit})$ can be identified with the natural projection

$$\mathrm{K}^b(h\mathcal{C}^{\heartsuit}) \to \mathrm{K}^b_w(h\mathcal{C}^{\heartsuit}).$$

Since t is weight-exact, it sends any choices of weight decompositions to weight decompositions in $\mathrm{Com}^b(h\mathcal{C}^{\heartsuit})$. Hence, by construction $t^{\mathrm{B}}_{h\mathcal{C}} \circ t \simeq t^{\mathrm{B}}_{\mathrm{K}^b(h\mathcal{C}^{\heartsuit})}$. □

Remark 3.2.11. The corollary above solves [Bon10, Conjecture 3.3.3] for triangulated categories with a bounded weight structure that have an ∞-categorical model.

Remark 3.2.12. For an additive ∞-category $\mathcal{A}$ we have a universal map to an n-category

$$\mathcal{A} \to \tau_{\leqslant n}\mathcal{A}.$$

The ∞-category $\tau_{\leqslant n}\mathcal{A}$ has the same objects as $\mathcal{A}$ and the mapping spaces in $\tau_{\leqslant n}\mathcal{A}$ are given by n-truncations of the mapping spaces in $\mathcal{A}$.

Under the equivalence Theorem 3.2.7 this gives some higher version of the weight complex functor $\mathcal{C} \to \tau_{\leqslant n}\mathcal{C} := (\tau_{\leqslant n}(\mathcal{C}^{\heartsuit}))^{\mathrm{fin}}$ for any boundedly weighted stable ∞-category $\mathcal{C}$. We refer to $\tau_{\leqslant n}\mathcal{C}$ as the n-th Postnikov truncation of $\mathcal{C}$.

3.3. **Examples.**

3.3.1. *Motivic examples.* In this section, we discuss examples of boundedly weighted stable ∞-categories coming from the theory of motives. We fix a field k and let R be a ring of coefficients. Denote by $\mathbf{DM}(k, \mathrm{R})$ the stable ∞-category of motives in the sense of Voevodsky, with coefficients in R (see [Voe00] for the original reference, [MVW06] for a textbook treatment and [BH18, Chapter 14] or [EK20] for ∞-categorical treatments). This ∞-category admits a functor

$$\mathrm{M} : \mathrm{Sm}_k \to \mathbf{DM}(k, \mathrm{R}),$$

which associates to a smooth k-scheme X, its motive $\mathrm{M}(\mathrm{X})$ and converts products of schemes into tensor products of motives; the coefficient ring will always be clear from the context. Inside $\mathbf{DM}(k, \mathrm{R})$ we have the subcategory

$$\mathbf{DM}_{\mathrm{gm}}(k, \mathrm{R}) \subset \mathbf{DM}(k, \mathrm{R})$$

of **geometric motives** [MVW06, Lecture 14]. This is the stable subcategory of $\mathbf{DM}(k, \mathrm{R})$, closed under retracts, generated by objects

$$\{\mathrm{M}(\mathrm{X})(q) : \mathrm{X} \text{ is a smooth } k\text{-scheme}, q \in \mathbb{Z}\}.$$

Definition 3.3.2. Let k be a field and R a ring of coefficients. The additive ∞-category of **Chow motives** (with coefficients in R):

$$\mathbf{Chow}_{\infty}(k, \mathrm{R}) \subset \mathbf{DM}_{\mathrm{gm}}(k, \mathrm{R}),$$

is the smallest additive ∞-category generated by

$$\{\mathrm{M}(\mathrm{X})(q)[2q] : \mathrm{X} \text{ is smooth and projective over } k, q \in \mathbb{Z}\},$$

and retracts thereof.

We note that the mapping spaces in $\mathbf{Chow}_{\infty}(k, \mathrm{R})$ are not discrete. Indeed, suppose that X is a smooth k-scheme and Y is a smooth projective k-scheme which is of pure dimension d. Then, if k is a *perfect field*, Friedlander-Voevodsky/Atiyah duality ([FV00, Rio05]) gives us a computation of the dual in $\mathbf{DM}(k; \mathrm{R})$

$$\mathrm{M}(\mathrm{Y})^{\vee} \simeq \mathrm{M}(\mathrm{Y})(-d)[-2d].$$

Therefore we have the following computation for all $j \geqslant 0$:

$$\begin{aligned}
\pi_j \operatorname{Maps}(\mathrm{M}(\mathrm{X})(n)[2n], \mathrm{M}(\mathrm{Y})(m)[2m]) &\simeq \pi_0 \operatorname{Maps}(\mathrm{M}(\mathrm{X}), \mathrm{M}(\mathrm{Y})(m-n)[2m-2n-j]) \\
&\simeq \pi_0 \operatorname{Maps}(\mathrm{M}(\mathrm{X}) \otimes \mathrm{M}(\mathrm{Y}), \mathrm{M}(k)(m-n+d)[2m-2n-j+2d]) \\
&\simeq \pi_0 \operatorname{Maps}(\mathrm{M}(\mathrm{X} \times \mathrm{Y}), \mathrm{M}(k)(m-n+d)[2m-2n-j+2d]) \\
&\simeq \mathrm{H}^{2(m+d-n)-j, m+d-n}_{\mathrm{mot}}(\mathrm{X} \times \mathrm{Y}) \\
&\simeq \mathrm{CH}^{m+d-n}(\mathrm{X} \times \mathrm{Y}; j).
\end{aligned}$$

This can be non-zero for $j > 0$. This calculation leads to the following lemma.

Lemma 3.3.3. *Let k be a perfect field,* X *be a smooth k-scheme,* Y *a smooth projective k-scheme of dimension d and $n, m \in \mathbb{Z}$. Then, for any coefficient ring* R*, the mapping spectrum*

$$\operatorname{maps}_{\mathbf{DM}_{\mathrm{gm}}(k;\mathrm{R})}(\mathrm{M}(\mathrm{X})(n)[2n], \mathrm{M}(\mathrm{Y})(m)[2m])$$

is connective.

Proof. By the computation above, we need to show that the group

$$\pi_j \operatorname{maps}(\mathrm{M}(\mathrm{X})(n)[2n], \mathrm{M}(\mathrm{Y})(m)[2m]) \simeq \mathrm{H}^{2(m+d-n)-j, m+d-n}_{\mathrm{mot}}(\mathrm{X} \times \mathrm{Y}; \mathrm{R}),$$

is zero whenever $j < 0$. Without loss of generality, we might as well assume that $\mathrm{R} = \mathbb{Z}$. By way of comparison with Bloch's higher Chow groups [Voe02]:

$$\mathrm{H}^{p,q}_{\mathrm{mot}}(\mathrm{X}; \mathbb{Z}) \simeq \mathrm{CH}^{q}(\mathrm{X}, 2q-p),$$

we see that the desired vanishing is true since $\mathrm{CH}^q(\mathrm{X}, i)$ is zero whenever $i < 0$. □

The following is a result of Bondarko which we give a proof of for the reader's convenience. We remark that, in what follows, R must be a ring in which the characteristic of k is invertible (e.g. $\mathbb{Z}[\frac{1}{e}]$). This lets us assume that k is perfect and, more importantly, allows us to use smooth *projective* k-schemes as generators for geometric motives.

Theorem 3.3.4 ([Bon10, Bon11])**.** *Let k be a field and suppose that e is the exponential characteristic of k. Assume that* R *is a ring where e is invertible. Then there exists a bounded weight structure on $\mathbf{DM}_{\mathrm{gm}}(k; \mathrm{R})$ such that $\mathbf{DM}_{\mathrm{gm}}(k, \mathrm{R})^{\heartsuit_w} \simeq \mathbf{Chow}_{\infty}(k, \mathrm{R})$.*

Proof. By [Sus17] we can assume k is perfect. By Remark 3.1.3 it suffices to check that the collection $\{\mathrm{M}(\mathrm{X})(q)[2q]\}$ where X is a smooth projective k-scheme and $q \in \mathbb{Z}$ is a collection of generators for $\mathbf{DM}_{\mathrm{gm}}(k, \mathrm{R})$ and that the mapping spectrum

$$\operatorname{maps}_{\mathbf{DM}_{\mathrm{gm}}(k;\mathrm{R})}(\mathrm{M}(\mathrm{X})(n)[2n], \mathrm{M}(\mathrm{Y})(m)[2m])$$

is connective for any smooth projective X, Y and any $n, m \in \mathbb{Z}$. The first fact follows from [Kel17, Proposition 5.3.3], the second fact was already proved in Lemma 3.3.3. $\square$

Remark 3.3.5. The analogous weight structure can be constructed more generally on the ∞-categories of relative geometric motives $\mathbf{DM}_{\mathrm{gm}}(S; \mathbb{Q})$ (see [Bon14]) for quasi-excellent finite dimensional separated schemes S and even on some subcategories of $\mathbf{DM}_{\mathrm{gm}}(S; \mathbb{Z}[1/p])$, where S is a scheme of exponential characteristic p (see [BI15]). Moreover, similar weight structures can be constructed on the ∞-category of compact KGL_S-modules and the ∞-category of compact MGL_S-modules ([BS18a, 4]).

3.3.6. *Stacky examples.* In this section, we will furnish some examples from the context of algebraic stacks. First we fix the definitions and introduce basic notions from the theory of spectral stacks.

Definition 3.3.7. A functor $F\colon \mathcal{E}_\infty\text{-Rings} \to \mathrm{Spc}$ is called a *spectral stack*[6] if the following holds:

- F is a sheaf in the fpqc topology on $\mathcal{E}_\infty\text{-Rings}^{op}$.
- there exists a surjection of sheaves $\mathrm{Spec}(R) \xrightarrow{f} F$, where Spec denotes the Yoneda embedding $\mathcal{E}_\infty\text{-Rings}^{op} \to \mathrm{Fun}(\mathcal{E}_\infty\text{-Rings}, \mathrm{Spc})$. We call this map an *atlas* of F.
- the map $\mathrm{Spec}(R) \to F$ of the map f is flat, that is for any map $\mathrm{Spec}(S) \to F$ the pullback map
$$\mathrm{Spec}(S) \times_F \mathrm{Spec}(R) \to \mathrm{Spec}(R)$$
is equivalent to $\mathrm{Spec}(f_R)$ for some flat map of $\mathcal{E}_\infty$-rings $R \xrightarrow{f_R} R'$.

We say that a spectral stack is connective if the ring spectrum R can be chosen to be connective.

Note that any spectral scheme can be viewed as a spectral stack via taking the associated representable functor. Denote by Stk_∞ the ∞-category of spectral stacks. We define $\mathbf{D}_{\mathrm{qc}}\colon \mathrm{Stk}_\infty \to \mathrm{Cat}_\infty^{ex}$ to be the right Kan extension of the functor $\mathbf{Mod}_R\colon \mathcal{E}_\infty\text{-Rings} \to \mathrm{Cat}_\infty^{ex}$. This functor admits a symmetric monoidal enhancement (see [Lur18, 6.2.6]). Moreover $\mathbf{D}_{\mathrm{qc},X}$ admits a t-structure where the nonnegative part is the full subcategory of connective objects, i.e., those objects that become connective after pullback to a flat atlas (see [Lur18, Corollary 9.1.3.2]). We refer to [Lur18, Chapter 9] for more generalities on spectral stacks.

Definition 3.3.8. Let k be a discrete commutative ring.

- A **group scheme** over k is a group object in the category of k-schemes. A group scheme is said to be flat if it is flat as a k-scheme.
- A flat group scheme G is called **embeddable** if there exists a homomorphism $G \to \mathrm{GL}_n(k)$ which is a closed immersion [AHR19, Definition 2.1(1)].
- A flat group scheme G is called **linearly reductive** if the functor of taking invariants
$$\mathbf{D}_{\mathrm{qc,BG}} \to \mathbf{D}_{\mathrm{qc},k}$$
is t-exact for the standard t-structure; this is the same as saying that the canonical morphism $\pi : \mathrm{BG} \to \mathrm{Spec}\, k$ is cohomologically affine in the sense of [Alp13].
- A flat group scheme G is called **nice** if it is an extension
$$1 \to G^0 \to G \to H \to 1;$$
where G^0 is closed, normal subgroup of multiplicative type (e.g. a torus) and H is finite, locally constant with $|H|$ invertible in k [HR15a, Definition 1.1].

[6]In [Lur18, Definition 9.1.0.1] the ∞-category Spc was enlarged to include spaces from a larger universe, i.e. not necesessarily small. This is not necessary for the definition; see [Lur18, Remark 9.1.0.2].

We refer to [AHR19, Section 2] for a more thorough exposition, but let us give some explanation and examples for the reader's convenience.

Example 3.3.9. It is standard that a constant, finite group with order invertible on the base is linearly reductive. Furthermore, according to [Alp13, Proposition 12.17(2)], linearly reductive groups are closed under extensions. Therefore any nice group scheme is linearly reductive.

Example 3.3.10. Let k be a $\mathbb{Q}$-algebra. Then, by [Alp14, Lemma 4.1.6, Remark 9.1.3], the notion of being linearly reductive is equivalent to the notion of being geometrically reductive (note that if G is linearly reductive then it is always geometrically reductive). This latter notion is equivalent to G being reductive in the classical sense whenever G is smooth with geometrically connected fibers [Alp14, Theorem 9.7.5]. Hence, in this setting, the reader should feel free to substitute the notion of linear reductivity with the classical notion of reductivity.

Example 3.3.11. Let k be a field of characteristic $p > 0$; from the perspective of topological cyclic homology this is the primary situation of interest. Then, being linearly reductive is equivalent to being nice [HR15a, Theorem 1.2]. Examples include tori, the roots of unity sheaves μ_{p^n} and constant group schemes of order prime to p. If k is a ring of mixed characteristic, then [AHR19, Theorem 19.9] gives a classification result: any linearly reductive group G over k is canonically an extension of a finite, tame linearly reductive group scheme by a smooth linearly reductive group scheme.

If X is a quasicompact and quasiseparated derived algebraic stack we recall $\mathbf{Perf}_{\mathrm{X}} \subset \mathbf{D}_{\mathrm{qc,X}}$ is the subcategory of spanned by those objects M such that for any morphism from an affine derived scheme $x : \operatorname{Spec} \mathrm{A} \to \mathrm{X}$, $x^*\mathrm{M}$ is a perfect A-module; such an object is a **perfect complex** on X.

Construction 3.3.12. We say that a group scheme G acts on $\mathcal{E}_\infty$-k-algebra R if R an $\mathcal{E}_\infty$-algebra object in $\mathbf{D}_{\mathrm{qc,BG}}$. Given an affine group scheme G over a field k acting on a connective $\mathcal{E}_\infty$-k-algebra R consider the functor

$$[\operatorname{Spec} \mathrm{R}/\mathrm{G}] \colon \mathcal{E}_\infty\text{- Rings} \to \mathrm{Spt}$$

which is the colimit of the simplicial diagram

$$\cdots \mathrm{G} \times_k \mathrm{G} \times_k \operatorname{Spec} \mathrm{R} \;\substack{\longrightarrow\\[-2pt]\longleftarrow\\[-2pt]\longrightarrow\\[-2pt]\longleftarrow\\[-2pt]\longrightarrow}\; \mathrm{G} \times_k \operatorname{Spec} \mathrm{R} \;\substack{\longrightarrow\\[-2pt]\longleftarrow\\[-2pt]\longrightarrow}\; \operatorname{Spec} \mathrm{R}$$

in Spt-valued fpqc-sheaves on $\mathcal{E}_\infty$-Rings^{op}. This turns out to be a stack and an atlas is given by $\operatorname{Spec} \mathrm{R} \to [\operatorname{Spec} \mathrm{R}/\mathrm{G}]$ (see [Lur18, Corollary 9.1.1.5]). We call it the *quotient stack*. When $\mathrm{R} = k$ and the action is trivial we adopt the notation $\mathrm{BG} := [\operatorname{Spec} k/\mathrm{G}]$. We have

$$\mathbf{D}_{\mathrm{qc},[\operatorname{Spec} \mathrm{R}/\mathrm{G}]} \simeq \lim \mathbf{D}_{\mathrm{qc},\mathrm{G}^{\times n} \times_k \operatorname{Spec} \mathrm{R}} \simeq \lim \mathbf{Mod}_{\mathcal{O}_{\mathrm{G}}^{\otimes n} \otimes_k \mathrm{R}}(\mathbf{D}_{\mathrm{qc},\mathrm{G}^{\times n}}) \simeq \mathbf{Mod}_{\mathrm{R}}(\mathbf{D}_{\mathrm{qc,BG}}),$$

where R is considered as an $\mathcal{E}_\infty$-algebra object in $\mathbf{D}_{\mathrm{qc,BG}}$ on the right-hand side. (cf. [Iwa18, Proposition 2.7]). Alternatively, we may identify the ∞-category $\mathbf{D}_{\mathrm{qc},[\operatorname{Spec} \mathrm{R}/\mathrm{G}]}$ with R-modules endowed with an action of G which is compatible with an action on R.

The next theorem is the main source of examples for this thesis.

Theorem 3.3.13. *Let* G *be an embeddable linearly reductive group scheme over a commutative ring* k.

(1) *Let* R *be a connective* $\mathbb{E}_\infty$-k-*algebra endowed with an action of* G. *There is a bounded weight structure on* $\mathbf{Perf}_{[\operatorname{Spec} \mathrm{R}/\mathrm{G}]}$ *whose heart is the full subcategory generated under retracts by the set*

$$\mathrm{N} := \{p^*\mathcal{E} : \mathcal{E} \textit{ is a finite } \mathrm{G}\textit{-equivariant } k\textit{-module } \mathcal{E}\},$$

where $p \colon [\operatorname{Spec} \mathrm{R}/\mathrm{G}] \to \mathrm{BG}$ *is the structure morphism.*

(2) *Let* $\mathrm{R} \xrightarrow{f} \mathrm{S}$ *be a* G*-equivariant morphism of connective* $\mathbb{E}_\infty$*-*k*-algebras endowed with an action of* G*. Then the base change functor* $\mathbf{Perf}_{[\mathrm{Spec\,R/G}]} \to \mathbf{Perf}_{[\mathrm{Spec\,S/G}]}$ *is weight exact.*

To prove this claim we will need to use that $\mathbf{D}_{\mathrm{qc},[\mathrm{Spec\,R/G}]}$ has a nice set of compact generators.

Lemma 3.3.14. *Let* $\mathcal{C}^\otimes$ *be a stable symmetric monoidal* ∞*-category whose underlying* ∞*-category is compactly generated by* $\{\mathrm{X}_i\}$ *and the unit is compact. Then for any* $\mathbb{E}_1$*-algebra object* A *in* $\mathcal{C}^\otimes$ *the* ∞*-category of left* A*-modules* $\mathbf{Mod}_{\mathrm{A}}(\mathcal{C})$ *is compactly generated by* $\{\mathrm{A} \otimes \mathrm{X}_i\}$*.*

Proof. Let $\mathrm{U} : \mathbf{Mod}_{\mathrm{A}}(\mathcal{C}) \to \mathcal{C}$ be the forgetful functor, right adjoint to $\mathrm{Y} \mapsto \mathrm{A} \otimes \mathrm{Y}$. By Corollaries 4.2.3.7(2) and 4.2.3.2 of [Lur17a], U preserves colimits and is conservative. This implies that the objects $\mathrm{A} \otimes \mathrm{X}_i$ are compact. It remains to show that they form a set of generators. If $\mathrm{Y} \in \mathcal{C}$ is an object such that the mapping space $\mathrm{Maps}_{\mathrm{A}}(\mathrm{A} \otimes \mathrm{X}_i, \mathrm{Y}[j]) = \mathrm{Maps}_{\mathcal{C}}(\mathrm{X}_i, \mathrm{U}(\mathrm{Y})[j])$ is contractible for all i and all $j \in \mathbf{Z}$, then $\mathrm{U}(\mathrm{Y}) = 0$. This implies that $\mathrm{Y} = 0$, whence the claim. □

Proposition 3.3.15. *Fix the notations of Theorem 3.3.13. The stable* ∞*-category* $\mathbf{D}_{\mathrm{qc},[\mathrm{Spec\,R/G}]}$ *is compactly generated by* N*. If* k *is a field it suffices to take the subset of* N *consisting of pullbacks of irreducible* k*-representations* $\mathrm{V} \in \mathbf{D}_{\mathrm{qc,BG}}$*.*

Proof. When $\mathrm{R} = k$, the claim follows from [HR15b, Proposition 8.4] as BG has the resolution property. Moreover, if k is a field then the ∞-category $\mathbf{D}_{\mathrm{qc,BG}}$ is compactly generated by the irreducible k-representations of G (see also [HR15a, Thm. A(c)]). Now the general statement follows from Lemma 3.3.14 and the discussion in the end of Construction 3.3.12. □

Proof of Theorem 3.3.13. We apply the construction from Remark 3.1.3 to the set N. It suffices to show that

(a) $\mathbf{Perf}_{[\mathrm{Spec\,R/G}]}$ is generated under finite limits, finite colimits and retracts by the set N,

(b) for any G-equivariant k-modules $\mathcal{E}$, $\mathcal{E}'$ the spectrum
$$\mathrm{maps}_{\mathbf{Perf}_{[\mathrm{Spec\,R/G}]}}(p^*\mathcal{E}, p^*\mathcal{E}')$$
is connective.

By Proposition 3.3.15 the ∞-category $\mathbf{D}_{\mathrm{qc},[\mathrm{Spec\,R/G}]}$ is compactly generated by N. In particular, this implies that the subcategory of compact objects in $\mathbf{D}_{\mathrm{qc},[\mathrm{Spec\,R/G}]}$ is generated by N under finite limits, finite colimits and retracts. To prove (a) it suffices to show that every perfect complex is compact in $\mathbf{D}_{\mathrm{qc},[\mathrm{Spec\,R/G}]}$. The set N contains the tensor unit of $\mathbf{D}_{\mathrm{qc},[\mathrm{Spec\,R/G}]}$, in particular the tensor unit is a compact object. Now the claim follows from [Lur18, Proposition 9.1.5.3] (see also [HR15b, Lemma 4.4(2)]). To prove (b) we use the equivalence
$$\mathrm{maps}_{\mathbf{Perf}_{[\mathrm{Spec\,R/G}]}}(p^*\mathcal{E}, p^*\mathcal{E}') \simeq \mathrm{maps}_{\mathbf{Perf}_{\mathrm{R}}}(p^*\mathcal{E}, p^*\mathcal{E}')^{\mathrm{G}}$$
given by descent and the fact that taking invariants preserves connectivity by definition of linear reductivity.

The second part of the statement follows by construction. (see also Remark 3.1.3). □

Remark 3.3.16. Under the assumptions in Theorem 3.3.13 $\mathbf{Perf}_{\mathrm{X}}$ coincides with compact objects in $\mathbf{D}_{\mathrm{qc,X}}$ [Lur18, Proposition 9.1.5.3] (see also [HR15b, Lemma 4.4(2)]). We note that a nice weight structure on $(\mathbf{D}_{\mathrm{qc,X}})^\omega$ exists in a more general setting. For instance, a slight modification of the formula (1) gives a generating set for a weight structure for G a finite (discrete) group and R any $\mathbb{E}_\infty$-ring spectrum. More precisely, one shall define
$$\mathrm{N} := \{p^*\mathcal{E} : \mathcal{E} \text{ is a finite } \textit{projective} \text{ G-equivariant } k\text{-module}\}.$$

Indeed, given the equivalence

$$(\mathbf{D}_{\mathrm{qc},[\mathrm{Spec\,R/G}]})^{\omega} \simeq (\mathbf{Mod}_{\mathrm{R}_{tw}[\mathrm{G}]})^{\omega} = \mathbf{Perf}_{\mathrm{R}_{tw}[\mathrm{G}]}$$

where $\mathrm{R_{tw}[G]}$ is the twisted group ring, this follows from Example 3.1.4. On the other hand,

$$(\mathbf{D}_{\mathrm{qc},[\mathrm{Spec\,R/G}]})^{\omega} \neq \mathbf{Perf}_{[\mathrm{Spec\,R/G}]},$$

so Theorem 3.3.13 can not be applied.

Unfortunately, we do not have a general unified statement about the existence of a weight structure on $(\mathbf{D}_{\mathrm{qc},[\mathrm{Spec\,R/G}]})^{\omega}$.

4. Regularity

4.1. **Regular ring spectra.** Suppose that R is a *left noetherian* $\mathbb{E}_1$-ring spectrum, i.e. that $\pi_0(\mathrm{R})$ is left noetherian as an ordinary ring and the homotopy groups $\pi_i(\mathrm{R})$ are finitely generated as left $\pi_0(\mathrm{R})$-modules. A left R-module M is *almost perfect* if it is bounded below and each $\pi_i(\mathrm{M})$ is finitely generated as a left $\pi_0(\mathrm{R})$-module [Lur17a, Prop. 7.2.4.17], and *coherent* if it is almost perfect and bounded above. By [Lur17a, Prop. 7.2.4.23(4)], an almost perfect R-module is perfect if and only if it is of finite tor-amplitude. We write $\mathbf{Coh}_{\mathrm{R}}$ for the full subcategory of $\mathbf{Mod}_{\mathrm{R}}$ spanned by coherent left R-modules; this is also a thick subcategory by [Lur17a, Prop. 7.2.4.11].

Remark 4.1.1. Note that the noetherian hypothesis guarantees that R is almost perfect as a left R-module. Thus if R is bounded above, then it is moreover coherent as a left R-module. It follows then that every perfect left R-module is coherent, i.e., that there is an inclusion $\mathbf{Perf}_{\mathrm{R}} \subset \mathbf{Coh}_{\mathrm{R}}$ when R is bounded above.

The following definition can be found in [BL14], where it is called "almost regularity".

Definition 4.1.2. Let R be a left noetherian connective $\mathbb{E}_1$-ring spectrum. We will say that R is *regular* if every coherent left R-module is perfect.

It suffices to require that every *discrete* coherent left R-module is perfect, in view of the exact triangles

$$\Sigma^i \pi_i(\mathrm{M}) \to \tau_{\leqslant i}(\mathrm{M}) \to \tau_{\leqslant i-1}(\mathrm{M})$$

for $\mathrm{M} \in \mathbf{Mod}_{\mathrm{R}}$ and $i \in \mathbf{Z}$. Moreover, one has the following useful criterion (see [BL14, Prop. 1.3]):

Proposition 4.1.3. *Let* R *be a left noetherian connective* $\mathbb{E}_1$*-ring such that* $\pi_0(\mathrm{R})$ *is a regular commutative ring. Then* R *is regular if and only if* $\pi_0(\mathrm{R})$ *is perfect as an* R*-module.*

Lemma 4.1.4. *Let* R *be a left noetherian connective* $\mathbb{E}_1$*-ring such that* $(\pi_0(\mathrm{R}), \mathbf{m})$ *is a local commutative ring. Suppose that for every discrete coherent right* R*-module* M*, the tensor product* $\mathrm{M} \otimes_{\mathrm{R}} \pi_0(\mathrm{R})/\mathbf{m}$ *is bounded. Then* R *is regular.*

Proof. It suffices to show that any discrete coherent right R-module M is of finite tor-amplitude. By our assumption, $\mathrm{M} \otimes \pi_0(\mathrm{R})/\mathbf{m}$ is k-truncated for some k. For any prime ideal $\mathbf{p} \subset \pi_0(\mathrm{R})$ there exists a minimal free resolution of the $\pi_0(\mathrm{R})/\mathbf{p}$-module $\tilde{\mathrm{M}} := \mathrm{M} \otimes_{\mathrm{R}} \pi_0(\mathrm{R})/\mathbf{p}$ by [Rob80, Theorem 2.4] (cf. [Ser68, Appendix I, Prop. 1(a)]). That is, there is a quasi-isomorphism $\mathrm{F} \to \tilde{\mathrm{M}}$ (where we regard $\tilde{\mathrm{M}}$ as a chain complex of discrete $\pi_0(\mathrm{R})/\mathbf{p}_i$-modules), where F is a complex of finite free $\pi_0(\mathrm{R})/\mathbf{p}_i$-modules whose differentials are defined by matrices with coefficients in $\mathbf{m}$. The homotopy groups

$$\pi_j\big(\mathrm{F} \otimes_{\pi_0(\mathrm{R})/\mathbf{p}} \pi_0(\mathrm{R})/\mathbf{m}\big) = \pi_j\big(\tilde{\mathrm{M}} \otimes_{\pi_0(\mathrm{R})/\mathbf{p}} \pi_0(\mathrm{R})/\mathbf{m}\big) = \pi_j\big(\mathrm{M} \otimes_{\mathrm{R}} \pi_0(\mathrm{R})/\mathbf{m}\big)$$

vanish for $j > k$, hence all terms of F must be zero in degrees higher than k. This implies that that $\mathrm{M} \otimes_{\mathrm{R}} \pi_0(\mathrm{R})/\mathbf{p}$ is k-truncated.

Now any discrete coherent left R-module N admits a filtration

$$0 = \mathrm{N}_0 \subset \cdots \subset \mathrm{N}_n = \mathrm{N}$$

such that the associated graded modules $\mathrm{N}_i/\mathrm{N}_{i-1}$ are of the form $\pi_0(\mathrm{R})/\mathbf{p}_i$ for some prime ideals $\mathbf{p}_i$. Considering the exact sequences

$$\mathrm{N}_{i-1} \to \mathrm{N}_i \to \pi_0(\mathrm{R})/\mathbf{p}_i$$

and using the fact that $\mathrm{M} \otimes_{\mathrm{R}} \pi_0(\mathrm{R})/\mathbf{p}_i$ is k-truncated for all i, we see that $\mathrm{M} \otimes \mathrm{N}$ is k-truncated as well. Thus M has finite tor-amplitude, as desired. □

Remark 4.1.5. Let R be a connective $\mathbb{E}_1$-ring. The ∞-category $\mathbf{Mod}_{\mathrm{R}}$ admits a canonical t-structure, where the non-negative part $(\mathbf{Mod}_{\mathrm{R}})_{t \geqslant 0}$ is spanned by the connective modules. If R is left noetherian this always restricts to the full subcategories of almost perfect and coherent modules [Lur17a, Prop. 7.2.4.18]. One might ask whether it also restricts to the full subcategory $\mathbf{Perf}_{\mathrm{R}}$ of perfect R-modules. This is the case if and only if $\pi_i(\mathrm{M})$ belongs to $\mathbf{Perf}_{\mathrm{R}}$ for any $\mathrm{M} \in \mathbf{Perf}_{\mathrm{R}}$. When R is left noetherian this is equivalent to R being regular.

Remark 4.1.6. For every $\mathrm{M} \in \mathbf{Perf}_{\mathrm{R}}$ and $\mathrm{N} \in \mathbf{Coh}_{\mathrm{R}}$, there exists an integer n such that the mapping space $\mathbf{Mod}_{\mathrm{R}}(\Sigma^n \mathrm{M}, \mathrm{N})$ is contractible. Indeed, this is true for $\mathrm{M} = \Sigma^j \mathrm{R}$ since N is bounded above, and therefore for an arbitary M, since it is built out of $\Sigma^j \mathrm{R}$ using finite colimits and retracts. For regular R, this says that the space $\mathbf{Mod}_{\mathrm{R}}(\mathrm{M}, \mathrm{N})$ has finitely many non-zero homotopy groups, for any $\mathrm{M}, \mathrm{N} \in \mathbf{Perf}_{\mathrm{R}}$.

Example 4.1.7. (i) If R is discrete, then it is regular if and only if it is regular in the sense of ordinary commutative algebra. This follows from Auslander-Buchsbaum-Serre theorem.

(ii) Let k be a regular commutative ring and $\mathrm{R} = k[\mathrm{T}]$ with T in degree 2. Then R is regular.

(iii) The $\mathcal{E}_\infty$-ring spectra ku, ko, and tmf are regular [BL14].

4.1.8. We say that R is *quasi-commutative* if, for every $x \in \pi_0(\mathrm{R})$ and $y \in \pi_n(\mathrm{R})$, the equality $xy = yx$ holds in $\pi_n(\mathrm{R})$. Under this assumption, we can show that regularity is stable under localizations:

Proposition 4.1.9. *Let* R *be a quasi-commutative connective* $\mathbb{E}_1$*-ring spectrum which is left noetherian.*

(i) *If* R *is regular, then for any set* $\mathrm{S} \subset \pi_0(\mathrm{R})$*, the localization* $\mathrm{S}^{-1}\mathrm{R}$ *is also regular.*

(ii) *If the localization* $\mathrm{R}_{\mathbf{m}}$ *is regular for every maximal ideal* $\mathbf{m} \subset \pi_0(\mathrm{R})$ *and furthermore* $\pi_*(\mathrm{R})$ *is a left noetherian graded ring, then* R *is regular.*

To prove the result we need two additional lemmas.

Lemma 4.1.10. *Let* R *be a quasi-commutative connective* $\mathbb{E}_1$*-ring spectrum which is left noetherian. Let* M *be an* R*-module and* $\mathbf{m}$ *be a maximal ideal in* $\pi_0(\mathrm{R})$*.*

Then any map $\mathrm{P}' \to \mathrm{M}_{\mathbf{m}}$ *in* $\mathbf{Mod}_{\mathrm{R}_{\mathbf{m}}}$ *with* $\mathrm{P}' \in \mathbf{Perf}_{\mathrm{R}_{\mathbf{m}}}$ *is equivalent to* $\mathrm{P}_{\mathbf{m}} \xrightarrow{f_{\mathbf{m}}} \mathrm{M}_{\mathbf{m}}$ *for some* $\mathrm{P} \in \mathbf{Perf}_{\mathrm{R}}$ *and* $f \in \mathbf{Mod}_{\mathrm{R}}(\mathrm{P}, \mathrm{M})$*.*

Proof. The functor

$$\mathbf{Mod}_{\mathrm{R}} \xrightarrow{(-)_{\mathbf{m}}} \mathbf{Mod}_{\mathrm{R}_{\mathbf{m}}}$$

is a localization whose kernel K is compactly generated (see [Lur17a, Lemma 7.2.3.13]). Hence, this functor restricts to a fully faithful embedding

$$\mathbf{Perf}_{\mathrm{R}}/(\mathrm{K} \cap \mathbf{Perf}_{\mathrm{R}}) \xrightarrow{\mathrm{L}} \mathbf{Perf}_{\mathrm{R}_{\mathbf{m}}}$$

which is an equivalence up to idempotent completion. Since $\pi_0(\mathrm{R})_{\mathbf{m}}$ is a local ring, $\mathrm{K}_0(\mathrm{R}_{\mathbf{m}}) \simeq \mathrm{K}_0(\pi_0(\mathrm{R}_{\mathbf{m}})) \simeq \mathbb{Z}$ and so the functor L induces an isomorphism on K_0. Now by [Tho97, Theorem 2.1] L is an equivalence and hence there exists $\mathrm{P}'' \in \mathbf{Perf}_{\mathrm{R}_{\mathbf{m}}}$ such that $\mathrm{P}''_{\mathbf{m}} \simeq \mathrm{P}'$.

Any morphism $f' \in \mathbf{Mod}_{\mathrm{R}_{\mathbf{m}}}(\mathrm{P}, \mathrm{M}_{\mathbf{m}}) \simeq \mathbf{Mod}_{\mathrm{R}}/\mathrm{K}(\mathrm{P}'', \mathrm{M})$ can be represented by a zig-zag of morphisms in $\mathbf{Mod}_{\mathrm{R}}$

$$\mathrm{P}'' \xleftarrow{s} \mathrm{P} \xrightarrow{f} \mathrm{M}$$

where $\mathrm{Cofib}(s) \in \mathrm{K}$. Since P'' is compact, and K is compactly generated we can assume $\mathrm{Cofib}(s)$ and P to be compact. This concludes the proof since by construction $f_{\mathbf{m}}$ is equivalent to f' and s becomes an isomorphism in $\mathbf{Mod}_{\mathrm{R}_{\mathbf{m}}}$. ☐

Lemma 4.1.11. *Let* R *be a connective* $\mathbb{E}_1$*-ring spectrum such that* $\pi_*(\mathrm{R})$ *is left noetherian graded ring. Then* $\pi_*(\mathrm{M})$ *is a finitely generated* $\pi_*(\mathrm{R})$*-module for any* $\mathrm{M} \in \mathbf{Perf}_{\mathrm{R}}$*.*

Proof. The claim is true for $\mathrm{M} = \Sigma^n \mathrm{R}$ for $n \in \mathbb{Z}$. For a fiber sequence $\mathrm{M}_1 \to \mathrm{M}_2 \to \mathrm{M}$ we have an exact sequence of graded $\pi_*(\mathrm{R})$-modules

$$\pi_*(\mathrm{M}_2) \to \pi_*(\mathrm{M}) \to \pi_{*-1}(\mathrm{M}_1).$$

Since $\pi_*(\mathrm{R})$ is noetherian, this implies that $\pi_*(\mathrm{M})$ is finitely generated whenever $\pi_*(\mathrm{M}_1)$ and $\pi_*(\mathrm{M}_2)$ are. Moreover, the property of having finitely generated π_* is closed under retracts, therefore it holds for all $\mathrm{M} \in \mathbf{Perf}_{\mathrm{R}}$ by definition of this category. ☐

proof of Proposition 4.1.9. For the purposes of the proof given a subset $\mathrm{S} \subset \pi_0(\mathrm{R})$ we denote the extension of scalars functor $\mathbf{Mod}_{\mathrm{R}} \to \mathbf{Mod}_{\mathrm{S}^{-1}\mathrm{R}}$ by L_{S}.

(i) It will suffice to show that, for every perfect $\mathrm{S}^{-1}\mathrm{R}$-module N, we have $\pi_0(\mathrm{N}) \in \mathbf{Perf}_{\mathrm{S}^{-1}\mathrm{R}}$. Since any perfect $\mathrm{S}^{-1}\mathrm{R}$-module N is a retract of an object of the form $\mathrm{L}_{\mathrm{S}}(\mathrm{M})$ for some $\mathrm{M} \in \mathbf{Perf}_{\mathrm{R}}$, we may restrict our attention to $\mathrm{N} = \mathrm{L}_{\mathrm{S}}(\mathrm{M})$. For $\mathrm{M} \in \mathbf{Perf}_{\mathrm{R}}$, it follows from [Lur17a, Prop. 7.2.3.20] that we have

$$\pi_n(\mathrm{L}_{\mathrm{S}}\mathrm{M}) \simeq \mathrm{S}^{-1}\pi_n(\mathrm{M}) \simeq \pi_n(\mathrm{M}) \otimes_{\pi_0(\mathrm{R})} \mathrm{S}^{-1}\pi_0(\mathrm{R}).$$

This implies in particular that $\pi_0(\mathrm{L}_{\mathrm{S}}\mathrm{M}) \simeq \mathrm{L}(\pi_0(\mathrm{M}))$ belongs to $\mathbf{Perf}_{\mathrm{S}^{-1}\mathrm{R}}$, as claimed.

(ii) Let M be a discrete coherent R-module. By the assumption $\mathrm{M}_{\mathbf{m}}$ is perfect for any maximal ideal $\mathbf{m} \subset \pi_0(\mathrm{R})$. Applying Lemma 4.1.10 to the identity map on $\mathrm{M}_{\mathbf{m}}$ we see that there is $\mathrm{P} \in \mathbf{Perf}_{\mathrm{R}}$ and a map $\mathrm{P} \xrightarrow{f} \mathrm{M}$ that becomes an equivalence after localizing at $\mathbf{m}$. For any $\mathrm{S} \subset \pi_0(\mathrm{R})$ the map of finitely generated $\pi_*(\mathrm{R})$-modules

$$\pi_*(\mathrm{P}) \xrightarrow{\pi_*(f)} \pi_*(\mathrm{M})$$

can be identified with

$$\pi_*(\mathrm{L}_{\mathrm{S}}\mathrm{P}) \xrightarrow{\pi_*(\mathrm{L}_{\mathrm{S}}f)} \pi_*(\mathrm{L}_{\mathrm{S}}\mathrm{M})$$

after localizing at S considered as a subset of $\pi_*(\mathrm{R})$ by [Lur17a, Prop. 7.2.3.20]. Therefore it becomes an isomorphism after inverting $(\pi_0(\mathrm{R}) \smallsetminus \mathbf{m}) \subset \pi_*(\mathrm{R})$. By Lemma 4.1.11 $\pi_*(\mathrm{P})$ is a finitely generated graded $\pi_*(\mathrm{R})$-module. The same is true for $\pi_*(\mathrm{M})$, since it is a finitely generated discrete $\pi_0(\mathrm{R})$-module. Hence we can find an element $x_{\mathbf{m}} \in \pi_0(\mathrm{R}) \smallsetminus \mathbf{m}$ such that

$$\pi_*(\mathrm{L}_{\{x_{\mathbf{m}}\}}\mathrm{P}) \xrightarrow{\pi_*(\mathrm{L}_{\{x_{\mathbf{m}}\}}f)} \pi_*(\mathrm{L}_{\{x_{\mathbf{m}}\}}\mathrm{M})$$

is also an isomorphism. Therefore $\mathrm{L}_{\{x_{\mathbf{m}}\}}\mathrm{M}$ is perfect. Now the ideal generated by $x_{\mathbf{m}}$ for all $\mathbf{m}$ is the unit ideal since it is not contained in any maximal ideal. Pick any finite set $\{x_{\mathrm{m}_1}, \cdots, x_{\mathrm{m}_n}\}$ that also generates the unit ideal. The family $\{\mathrm{R} \to \mathrm{R}_{x_{\mathrm{m}_i}}\}_{i=1}^n$ is a fpqc cover, so by [Lur18, Proposition 2.8.4.2(10)] M is perfect. ☐

4.1.12. *Bounded regular ring spectra.* This subsection is dedicated to the following result, which was previously noted for dg-algebras by Jørgensen in [Jø10, Theorem A] and for $\mathcal{E}_\infty$-ring spectra by Lurie in [Lur18, Lemma 11.3.3.3]. Here we only note that the same proof works more generally for quasi-commutative $\mathbb{E}_1$-ring spectra. Note that this is optimal in the sense that the statement is false without the quasi-commutativity assumption (see §4.1.14).

Theorem 4.1.13. *Let* R *be a quasi-commutative connective* $\mathbb{E}_1$*-ring spectrum which is left noetherian. If* R *is regular and has only finitely many nontrivial homotopy groups, then* R *is discrete.*

Proof. We can clearly assume that R is nonzero. To show that R is discrete, it will suffice to show that its localization $\mathrm{R}_{\mathbf{p}}$, at any prime ideal $\mathbf{p} \subset \pi_0(\mathrm{R})$, is discrete. Since $\mathrm{R}_{\mathbf{p}}$ is regular by Proposition 4.1.9, we may replace R by $\mathrm{R}_{\mathbf{p}}$ and thereby assume that $\pi_0(\mathrm{R})$ is local. Denote by $\mathbf{m} \subset \pi_0(\mathrm{R})$ the maximal ideal, and κ the residue field. Let $n \geqslant 0$ be the largest integer such that $\pi_n(\mathrm{R}) \neq 0$. For the sake of contradiction, we will assume that $n > 0$.

Since R is noetherian, the $\pi_0(\mathrm{R})$-module $\pi_n(\mathrm{R})$ is finitely generated. Let $\mathbf{p} \subset \pi_0(\mathrm{R})$ be a minimal prime ideal such that $\mathbf{p} \in \mathrm{Supp}_{\pi_0(\mathrm{R})}(\pi_n(\mathrm{R}))$. By Proposition 4.1.9, the localization $\mathrm{R}_{\mathbf{p}}$ is regular. Therefore, replacing R by $\mathrm{R}_{\mathbf{p}}$, we may assume that $\pi_n(\mathrm{R})$ is supported at $\mathbf{m}$, so that in particular $\mathbf{m} \subset \pi_0(\mathrm{R})$ is an associated prime ideal of $\pi_n(\mathrm{R})$.

Since R is regular, the coherent left R-module κ is perfect. Therefore, the following claim will yield the desired contradiction:

($*$) Let N be a nonzero connective perfect left R-module. Let $k \geqslant 0$ be the smallest natural number such that N is of tor-amplitude $\leqslant k$. Then we have $\pi_{n+k}(\mathrm{N}) \neq 0$.

To prove ($*$) we argue by induction. In the case $k = 0$, N is flat with $\pi_0(\mathrm{N}) \neq 0$, so that $\pi_n(\mathrm{N}) \simeq \pi_n(\mathrm{R}) \otimes_{\pi_0(\mathrm{R})} \pi_0(\mathrm{N}) \neq 0$. Assume therefore that $k > 0$. Choose elements of $\pi_0(\mathrm{N})$ that induce a basis of the κ-vector space $\pi_0(\mathrm{N}) \otimes_{\pi_0(\mathrm{R})} \kappa$ and consider the surjective R-module morphism $u : \mathrm{R}^{\oplus m} \to \mathrm{N}$ they determine. Its fiber, which we denote N', is a connective perfect left R-module fitting in an exact triangle

$$\mathrm{N}' \xrightarrow{v} \mathrm{R}^{\oplus m} \xrightarrow{u} \mathrm{N}.$$

Considering the induced exact sequence of abelian groups

$$0 = \pi_{n+k}(\mathrm{R}^{\oplus m}) \to \pi_{n+k}(\mathrm{N}) \to \pi_{n+k-1}(\mathrm{N}') \xrightarrow{\varphi} \pi_{n+k-1}(\mathrm{R}^{\oplus m}),$$

it will suffice to show that φ is not injective.

Since N is of tor-amplitude $\leqslant k$, with k minimal and > 0, it follows that N' is nonzero and of tor-amplitude $\leqslant k-1$. Therefore by the inductive hypothesis, $\pi_{n+k-1}(\mathrm{N}') \neq 0$. If $k > 1$, then $\pi_{n+k-1}(\mathrm{R}^{\oplus m}) = 0$, so φ is not injective. It remains to consider the case $k = 1$. In this case, the perfect left R-module N' is flat, and hence free of finite rank $r \geqslant 0$, since $\pi_0(\mathrm{R})$ is local. Therefore the map $v : \mathrm{N}' \to \mathrm{R}^{\oplus m}$ is determined up to homotopy by a matrix $\{v_{i,j}\}_{1 \leqslant i \leqslant r, 1 \leqslant j \leqslant m}$ of elements $v_{i,j} \in \pi_0(\mathrm{R})$. Since $\mathbf{m} \subset \pi_0(\mathrm{R})$ is an associated prime of $\pi_n(\mathrm{R})$, we may choose a nonzero element $x \in \pi_n(\mathrm{R})$ such that multiplication by x kills $\mathbf{m}$. We claim that the element $(x, x, \ldots, x) \in \pi_n(\mathrm{R})^{\oplus r} \simeq \pi_n(\mathrm{N}')$ is sent by φ to zero in $\pi_n(\mathrm{R}^{\oplus m})$. For this purpose it will suffice to show that the elements $v_{i,j} \in \pi_0(\mathrm{R})$ all belong to $\mathbf{m}$. Consider the exact sequence

$$\pi_0(\mathrm{N}')/\mathbf{m} \xrightarrow{v} \pi_0(\mathrm{R}^{\oplus m})/\mathbf{m} \xrightarrow{u} \pi_0(\mathrm{N})/\mathbf{m} \to 0.$$

Since the map $u : \pi_0(\mathrm{R}^{\oplus m})/\mathbf{m} \to \pi_0(\mathrm{N})/\mathbf{m}$ is injective, it follows that $v : \pi_0(\mathrm{N}')/\mathbf{m} \to \pi_0(\mathrm{R}^{\oplus m})/\mathbf{m}$ is the zero map. In other words, the image of $v : \pi_0(\mathrm{N}') \to \pi_0(\mathrm{R}^{\oplus m})$ lands in $\mathbf{m}^{\oplus m} \subset \pi_0(\mathrm{R})^{\oplus m}$, as desired. □

4.1.14. *A noncommutative counterexample.* We now give an example of a *noncommutative* regular ring spectrum that is bounded but not discrete. This shows that the quasi-commutativity hypothesis in Theorem 4.1.13 is necessary.

Construction 4.1.15. Let k be a field. Consider the graded ring R, concentrated in degrees 0 and 1, where $\mathrm{R}_0 = k \times k$ and $\mathrm{R}_1 = k$ with multiplication defined by the formulas

$$(a, b) \cdot (c, d) = (ac, bd)$$
$$x \cdot (a, b) = bx$$
$$(a, b) \cdot x = ax,$$

where $(a, b), (c, d) \in \mathrm{R}_0$, $x \in \mathrm{R}_1$.

The graded ring R is associative but not commutative, and gives rise to a connective $\mathbb{E}_1$-ring spectrum, which we denote again by R, that is not quasi-commutative. However, it has regular π_0, and is also itself regular:

Proposition 4.1.16. *Let* R *be the* $\mathbb{E}_1$*-ring spectrum defined by Construction 4.1.15. Then* R *is regular.*

Proof. By Proposition 4.1.3 it suffices to show that $\pi_0(\mathrm{R})$ is perfect.

Denote by X the direct summand of the free R-module R corresponding to the projector $(0, 1) \in \pi_0(\mathrm{R}) = k \times k$. We first note that $\mathrm{X} \simeq k$ is discrete. Indeed we have by definition $\pi_0(\mathrm{X}) = k$ and $\pi_i(\mathrm{X}) = 0$ for $i \neq 0, 1$. The only $x \in k$ satisfying $(0, 1) \cdot x = x$ is zero, so $\pi_1(\mathrm{X})$ is also zero.

Since $x \cdot (0, 1) = x \in \pi_1(\mathrm{R})$, the element $1 \in k \simeq \pi_1(\mathrm{R})$ gives rise to a map $\mathrm{X}[1] \to \mathrm{R}$. This map induces an isomorphism $\pi_0(\mathrm{X}) \to \pi_1(\mathrm{R})$, and therefore exhibits $\mathrm{X}[1]$ as the 1-connective cover $\tau_{\geqslant 1}\mathrm{R}$. It follows that its cofiber is an object of $\mathbf{Perf}_{\mathrm{R}}$ equivalent to $\tau_{\leqslant 0}\mathrm{R} = \pi_0(\mathrm{R})$. □

Remark 4.1.17. In fact it is easy to see using the Künneth spectral sequence that any $\mathbb{E}_1$-ring spectrum R is regular as soon as the classical ring $\pi_*(\mathrm{R}) = \bigoplus_n \pi_n(\mathrm{R})$ is regular. In the example above, the ring $\pi_*(\mathrm{R})$ is isomorphic to the ring of triangular matrices, which is regular.

4.2. **Regular spectral stacks.**

Definition 4.2.1.

(i) We say that $\mathrm{X} \in \mathrm{Stk}_\infty$ is *homologically regular* if the t-structure on $\mathbf{D}_{\mathrm{qc},\mathrm{X}}$ restricts to the subcategory of compact objects, or in other words, if $\pi_0(\mathrm{P})$ is compact for any compact P.

(ii) We say that $\mathrm{X} \in \mathrm{Stk}_\infty$ is *regular* if there exists a smooth, faithfully flat morphism $\mathrm{Spec}(\mathrm{R}) \to \mathrm{X}$ with R a regular connective $\mathcal{E}_\infty$-ring spectrum.

The two definitions are not in general equivalent because homological regularity is not local (even in the étale topology). Indeed, for a field k of positive characteristic p and a finite discrete group G of order p, the quotient stack $[\mathrm{Spec}(k)/\mathrm{G}]$ is not homologically regular.

However, for $\mathrm{X} = \mathrm{Spec}(\mathrm{R})$, where R is some connective $\mathcal{E}_\infty$-ring spectrum, the two notions are equivalent. Indeed, (i) implies (ii) tautologically, while the converse implication follows from [Lur18, Proposition 2.8.4.2]. Now we compare the two notions for more general spectral stacks.

4.2.2. *Comparing the two notions of regularity.* For the rest of the section, we fix a field k, G an embeddable group scheme over k, and R a noetherian connective $\mathbb{E}_\infty$-k-algebra with an action of G such that $\pi_0(\mathrm{R})$ is a finite type k-algebra. We additionally assume that either $char(k) = 0$ or G is linearly reductive. We write $\mathrm{X} = [\mathrm{Spec}(\mathrm{R})/\mathrm{G}]$ and $f\colon \mathrm{X} \to \mathrm{BG}$, $p\colon \mathrm{Spec}\,\mathrm{R} \to \mathrm{X}$

for the canonical morphisms. Under the assumptions the functor $f^*\colon \mathbf{D}_{\mathrm{qc,BG}} \to \mathbf{D}_{\mathrm{qc,X}} \simeq \mathbf{Mod}_{\mathrm{R}}(\mathbf{D}_{\mathrm{qc,BG}})$ can be identified with $\mathrm{V} \mapsto \mathrm{V} \otimes \mathrm{R}$. The inverse image functor

$$p^*\colon \mathbf{D}_{\mathrm{qc,X}} \to \mathbf{Mod}_{\mathrm{R}},$$

is identified with the functor of forgetting the action. The main theorem of this section is:

Theorem 4.2.3. *The spectral stack* $[\mathrm{Spec}(\mathrm{R})/\mathrm{G}]$ *is homologically regular if* R *is regular. Moreover, the converse implication holds if* G *is smooth and* $\pi_*(\mathrm{R})$ *is graded noetherian.*

In particular, we see that for the spectral stack $[\mathrm{Spec}(\mathrm{R})/\mathrm{G}]$ with smooth G and noetherian $\pi_*(\mathrm{R})$, homological regularity coincides with regularity. Note that the graded ring $\pi_*(\mathrm{R})$ is automatically noetherian if R is a bounded and noetherian $\mathbb{E}_\infty$-ring spectrum. Combining this with Theorem 4.1.13, we get:

Corollary 4.2.4. *Assume that* G *is smooth. If* R *is bounded and the spectral stack* $[\mathrm{Spec}\,\mathrm{R}/\mathrm{G}]$ *is homologically regular, then* R *is discrete.*

Proof of Theorem 4.2.3. First, assume that R is regular. Recall that by Theorem 3.3.13 To show X is homologically regular it suffices to show that for each of the compact generators $f^*(\mathrm{V})$ from Proposition 3.3.15, $\pi_n(f^*(\mathrm{V}))$ is compact for all n. By t-exactness of p^* (see [Lur18, Remark 9.1.3.4]), we have $p^*\pi_n(f^*(\mathrm{V})) = \pi_n(p^*f^*(\mathrm{V}))$. Since $p^*f^*(\mathrm{V}) = \mathrm{R} \otimes_k \mathrm{V}$ is a free R-module, it is a compact object of $\mathbf{Mod}_{\mathrm{R}}$. If R is regular, then $\pi_n(p^*f^*(\mathrm{V}))$ is compact, so it follows from [Lur18, Prop. 9.1.5.3] and compactness of $\mathcal{O}_{\mathrm{X}} = f^*(k)$ that $\pi_n(f^*(\mathrm{V}))$ is also compact. Thus X is homologically regular.

Now we prove the 'moreover' part. Assume R is not regular. We will argue by contradiction that X cannot be homologically regular. By Proposition 4.1.9 we know that $\mathrm{R}_{\mathbf{m}}$ is not regular for some maximal ideal $\mathbf{m} \subset \pi_0(\mathrm{R})$. By Lemma 4.1.4 there exists a discrete coherent R-module M such that $\mathrm{M}_{\mathbf{m}} \otimes_{\mathrm{R}_{\mathbf{m}}} \pi_0(\mathrm{R}_{\mathbf{m}})/\mathbf{m}$ has infinitely many nonzero homotopy groups. It follows that $\mathrm{M} \otimes_{\mathrm{R}} \pi_0(\mathrm{R})/\mathbf{m}$ is also unbounded. By [Mil13, Prop. 7.4] and the smoothness assumption the G-orbit of the point of $\mathrm{Spec}(\pi_0(\mathrm{R}))$ corresponding to the ideal $\mathbf{m}$ is regular. Therefore, if $\mathrm{I} \subset \pi_0(\mathrm{R})$ denotes the ideal corresponding to the closure of the orbit, then there is an element $f \notin \mathbf{m}$ such that $(\pi_0(\mathrm{R})/\mathrm{I})_f$ is regular. Now $\pi_0(\mathrm{R})/\mathbf{m}$ has a structure of a $(\pi_0(\mathrm{R})/\mathrm{I})_f$-module and we have an equivalence

$$\mathrm{M} \otimes_{\mathrm{R}} \pi_0(\mathrm{R})/\mathbf{m} \simeq \big(\mathrm{M} \otimes_{\mathrm{R}} (\pi_0(\mathrm{R})/\mathrm{I})_f\big) \otimes_{(\pi_0(\mathrm{R})/\mathrm{I})_f} \pi_0(\mathrm{R})/\mathbf{m}.$$

In particular, the right-hand side is unbounded. But since $(\pi_0(\mathrm{R})/\mathrm{I})_f$ is regular, this implies that the $(\pi_0(\mathrm{R})/\mathrm{I})_f$-module

$$\mathrm{M} \otimes_{\mathrm{R}} (\pi_0(\mathrm{R})/\mathrm{I})_f \simeq (\mathrm{M} \otimes_{\mathrm{R}} \pi_0(\mathrm{R})/\mathrm{I})_f$$

is unbounded and hence so is $\mathrm{M} \otimes_{\mathrm{R}} \pi_0(\mathrm{R})/\mathrm{I}$. In particular, $\pi_0(\mathrm{R})/\mathrm{I}$ cannot be perfect as an R-module.

Since the ideal I is G-invariant, $\pi_0(\mathrm{R})/\mathrm{I}$ comes as $p^*(\mathrm{M})$ for some $\mathrm{M} \in \mathbf{D}_{\mathrm{qc,X}}$. It follows from Proposition 3.3.15 that $f^*\mathrm{V}$ is a compact object of $\mathbf{D}_{\mathrm{qc,X}}$ for any finite dimensional k-representation V of G. If X is homologically regular, $\pi_0(f^*(\mathrm{V}))$ is then also a compact object in $\mathbf{D}_{\mathrm{qc,X}}$. The same holds for M, since it can be written as π_0 of the homotopy cofibre of a G-equivariant R-linear map $\varphi : \mathrm{V} \otimes_k \pi_0(\mathrm{R}) \to \pi_0(\mathrm{R})$ (V is any finite-dimensional subrepresentation of $\pi_0(\mathrm{R})$ containing a finite set of generators of I). But then the R-module $\pi_0(\mathrm{R})/\mathrm{I}$ is compact (= perfect), since the functor p^* preserves compact objects, whence the desired contradiction. □

Remark 4.2.5. Theorem 4.2.3 gives a lot of examples of homologically regular spectral stacks. In fact, its proof shows that a spectral quotient stack $\mathrm{X} = [\mathrm{Spec}\,\mathrm{R}/\mathrm{G}]$, defined over any ring k, is homologically regular, provided R is a regular $\mathcal{E}_\infty$-ring and G is a finite group whose order is invertible in k.

Thus we see that $[\mathrm{Spec}(\mathrm{ku}[1/2])/\mathrm{C}_2]$ is a homologically regular spectral stack for the canonical action of C_2. Arguing as in [BL14] one can show that there is a fiber sequence

$$\mathrm{K}^{\mathrm{C}_2}(\mathbf{Z}[1/2]) \to \mathrm{K}^{\mathrm{C}_2}(\mathrm{ku}[1/2]) \to \mathrm{K}^{\mathrm{C}_2}(\mathrm{KU}[1/2]).$$

4.3. **Adjacent structures.**

Definition 4.3.1. Let $\mathcal{C}^{\otimes}$ be a stable symmetric monoidal ∞-category such that the underlying ∞-category is endowed with a weight structure w. We say that the weight structure is *compatible* with the symmetric monoidal structure if $\mathcal{C}_{w\geqslant 0}$ and $\mathcal{C}_{w\leqslant 0}$ are closed under the tensor product operations.

In this situation the symmetric monoidal structure restricts to the subcategory $\mathcal{C}^{\heartsuit}$. We denote the corresponding symmetric monoidal ∞-category by $\mathcal{C}^{\heartsuit,\otimes}$.

4.3.2. We already know from Theorem 3.2.3 that stable ∞-categories together with a bounded weight structure correspond to additive ∞-categories, at least up to idempotent completion. It was shown in [Aok19, Lemma 4.2] that compatible weight structures correspond to symmetric monoidal additive ∞-categories in the same fashion:

Proposition 4.3.3. *Let $\mathcal{C}^{\otimes}$ be a stable idempotent complete symmetric monoidal ∞-category whose underlying ∞-category is endowed with a compatible bounded weight structure. Then there is a weight-exact equivalence of symmetric monoidal ∞-categories $\mathcal{C}^{\otimes} \cong (\mathcal{C}^{\heartsuit})^{\mathrm{fin},\otimes}$, where $(\mathcal{C}^{\heartsuit})^{\mathrm{fin},\otimes}$ is a natural symmetric monoidal refinement of the construction from Section 3.2.*

Example 4.3.4. Let R be a connective $\mathbb{E}_{\infty}$-ring spectrum. The weight structure on $\mathbf{Perf}_{\mathrm{R}}$ from Example 3.1.4 is compatible with the canonical symmetric monoidal structure.

As we have already explained in Section 3 weight structures and t-structures behave completely differently. For instance, there is no simple characterization of t-structures analogous to Theorem 3.2.3. They are however related via the following result due to Bondarko [Bon18, Theorem 0.1].

Theorem 4.3.5. *Let $\mathcal{C}$ be a stable ∞-category that admits all small coproducts and satisfies the Brown representability theorem (i.e., every coproduct preserving homological functor $\mathcal{C}^{op} \to \mathrm{A}b$ is representable). For instance, $\mathcal{C}$ is compactly generated. Suppose $(\mathcal{C}_{w\geqslant 0}, \mathcal{C}_{w\leqslant 0})$ is a weight structure such that $\mathcal{C}_{w\geqslant 0}$ is closed under arbitrary coproducts. Then there exists a t-structure on $\mathcal{C}$ such that $\mathcal{C}_{t\geqslant 0} = \mathcal{C}_{w\geqslant 0}$.*

When $\mathcal{C}$ does not have all coproducts, such a t-structure may not exist. When it does, it is called an adjacent t-structure:

Definition 4.3.6. Let $\mathcal{C}$ be a stable ∞-category. Given a weight structure $(\mathcal{C}_{w\geqslant 0}, \mathcal{C}_{w\leqslant 0})$ and a t-structure $(\mathcal{C}_{t\geqslant 0}, \mathcal{C}_{t\leqslant 0})$, we say that they are *adjacent* if the equality $\mathcal{C}_{t\geqslant 0} = \mathcal{C}_{w\geqslant 0}$ holds.

We write Ht for the heart of the t-structure.

Remark 4.3.7. Let $\mathcal{C}$ be a stable ∞-category with a weight structure. If an adjacent t-structure exists, then it is completely determined as follows: $\mathcal{C}_{t\geqslant 0} = \mathcal{C}_{w\geqslant 0}$, and $\mathcal{C}_{t\leqslant 0}$ is the full subcategory of objects $\mathrm{X} \in \mathcal{C}$ such that $\pi_0 \mathrm{Maps}_{\mathcal{C}}(\Sigma \mathrm{Y}, \mathrm{X}) = 0$ for all $\mathrm{Y} \in \mathcal{C}_{w\geqslant 0}$. In other words, the existence of an adjacent t-structure is a property (as opposed to additional structure).

Example 4.3.8. Let R be a regular $\mathbb{E}_1$-ring spectrum and consider the ∞-category $\mathbf{Perf}_{\mathrm{R}}$. Its canonical t- and weight structures (Remark 4.1.5 and Example 3.1.4) are adjacent. Conversely, any non-regular R provides an example of a weight structure with no adjacent t-structure.

4.3.9. Let $\mathcal{C}$ be a monoidal stable ∞-category with a bounded weight structure. We think of $\mathcal{C}$ as a ring spectrum with "many objects". In view of Example 4.3.8, the existence of an adjacent t-structure can be viewed as a regularity condition on $\mathcal{C}$. If $\mathcal{C}$ is moreover symmetric monoidal, in a way compatible with the weight structure (in the sense of Definition 4.3.1), then we think of $\mathcal{C}$ as a *commutative* regular ring spectrum with many objects.

With this in mind we expect the following generalization of Theorem 4.1.13:

Conjecture 4.3.10. *Let $\mathcal{C}^{\otimes}$ be a stable idempotent complete symmetric monoidal ∞-category with a compatible bounded weight structure w (in the sense of Definition 4.3.1). Suppose there exists a natural number N such that $\pi_n\mathcal{C}^{\heartsuit}(\mathrm{X},\mathrm{Y}) = 0$ for all $n \geqslant \mathrm{N}$ and $\mathrm{X},\mathrm{Y} \in \mathcal{C}^{\heartsuit}$. If w admits an adjacent t-structure, then $\mathcal{C}^{\heartsuit}$ is discrete.*

In particular the conjecture implies that under the conditions, $\mathcal{C}$ is equivalent to the ∞-category of bounded complexes over $\mathcal{C}^{\heartsuit}$ and also to the derived ∞-category of Ht.

In fact, we will see that this conjecture not only generalizes Theorem 4.1.13 but also Corollary 4.2.4.

Remark 4.3.11. Since $(\mathrm{X})_{w\geqslant 0}$ is spanned by the connective objects, the weight structure is adjacent to the canonical t-structure. The induced weight structure on the subcategory of compact objects admits an adjacent t-structure if and only if the canonical t-structure restricts to the subcategory of compact objects, in other words, if and only if X is homologically regular (see Definition 4.2.1). Moreover, R is bounded if and only if there exists an $\mathrm{N} > 0$ such that $\pi_n \operatorname{Maps}_{\mathbf{D}_{\mathrm{qc},\mathrm{X}}}(\mathrm{P},\mathrm{Q}) = 0$ for all $n \geqslant \mathrm{N}$ and all P,Q from the heart of the weight structure. Indeed, if R is bounded then for all irreducible G-representations $\mathrm{V}_i, \mathrm{V}_j$ we see that

$$\operatorname{Maps}_{\mathbf{D}_{\mathrm{qc},\mathrm{X}}}(\mathrm{R}\otimes \mathrm{V}_i, \mathrm{R}\otimes \mathrm{V}_j) = \operatorname{Maps}_{\mathbf{D}_{\mathrm{qc},\mathrm{BG}}}(\mathrm{V}_i, \mathrm{R}\otimes \mathrm{V}_j) = \underline{\operatorname{Hom}}_k(\mathrm{V}_i, \mathrm{R}\otimes_k \mathrm{V}_j)^{\mathrm{G}}$$

is bounded, since the functor of G-invariants is exact. Since $\mathrm{R}\otimes \mathrm{V}_i$ generate the heart of the weight structure, the same holds for $\operatorname{Maps}_{\mathbf{D}_{\mathrm{qc},\mathrm{X}}}(\mathrm{P},\mathrm{Q})$ with P,Q arbitrary objects in the heart of the weight structure. Conversely, suppose there is an integer N such that for all $n \geqslant \mathrm{N}$, $\pi_n \operatorname{Maps}_{\mathbf{D}_{\mathrm{qc},\mathrm{X}}}(\mathrm{R}\otimes \mathrm{V}_i, \mathrm{R}\otimes k) = 0$ for all irreducible representations V_i, where k is the trivial representation. Then by adjunction it follows that $\mathrm{R} \in \mathbf{D}_{\mathrm{qc},\mathrm{BG}}$ is bounded (with respect to the canonical t-structure). In particular, its underlying k-module is bounded.

By Remark 4.3.11, the conditions of Conjecture 4.3.10 are satisfied for $\mathbf{D}_{\mathrm{qc},\mathrm{X}}$. In this case, the conjecture is verified by Corollary 4.2.4. That is, we have:

Example 4.3.12. Conjecture 4.3.10 holds for $\mathcal{C}$ under the following additional assumption:

(∗) There is a symmetric monoidal weight-exact equivalence of stable ∞-categories

$$\mathcal{C} \simeq \mathbf{D}_{\mathrm{qc},[\mathrm{Spec}(\mathrm{R})/\mathrm{G}]},$$

where G is a linearly reductive group scheme of finite type over a field k, and R is a noetherian $\mathcal{E}_\infty$-k-algebra with G-action such that $\pi_0(\mathrm{R})$ is a finite type k-algebra.

Remark 4.3.13. In characteristic zero, the Tannaka formalism of Iwanari [Iwa18, Thm. 4.1, Prop. 4.9] gives an intrinsic characterization of $\mathcal{C}$ satisfying the condition (∗). Namely, it is equivalent to the following two conditions:

(1) $\mathcal{C}^{\heartsuit}$ is generated as a symmetric monoidal additive ∞-category by a wedge-finite object C of dimension d and its dual $\mathrm{C}^{\vee}$. Recall that C is *wedge-finite* of dimension $d \geqslant 0$ if $\Lambda^{d+1}(\mathrm{C}) = 0$ and $\Lambda^{d}(\mathrm{C})$ is $\otimes$-invertible. Any wedge-finite object C is dualizable, and we write $\mathrm{C}^{\vee}$ for its dual. See [Iwa18, Def. 1.1, Rem. 1.4].

(2) The $\mathcal{E}_\infty$-ring spectrum

$$\mathrm{A} = \bigoplus_{\lambda\in\mathbb{Z}^d_\star} \mathcal{C}^{\heartsuit}(\mathrm{S}_\lambda\mathrm{C}, \mathbf{1}) \otimes_k \mathrm{S}_\lambda\mathrm{K}$$

is noetherian and $\pi_0(\mathrm{A})$ is of finite type over k. Here $\mathbf{Z}^d_\star$ is the set of d-tuples

$$(\lambda_1, \cdots, \lambda_d) \in \mathbf{Z}^d$$

such that $\lambda_1 \geqslant \cdots \geqslant \lambda_d$, S_λ is the Schur functor corresponding to λ, and K is the regular representation of GL_d.

4.4. **Gluing weight structures.** As we have seen in Section 4.1.14 there exist bounded regular $\mathbb{E}_1$-rings that are not discrete, given by some triangular matrix rings. In particular this shows that Conjecture 4.3.10 could fail for weight structures that are not compatible with any symmetric monoidal structure. It turns out that the conjecture in such generality fails consistently: we prove a general gluing result for adjacent structures which gives us plenty of counterexamples to the conjecture as a byproduct. For instance, it implies the following statement:

Corollary 4.4.1. *The triangular matrix $\mathbb{E}_\infty$-ring spectrum*

$$\mathrm{T} = \begin{pmatrix} \mathrm{R} & \mathrm{M} \\ 0 & \mathrm{S} \end{pmatrix}$$

is regular, where R *and* S *are regular $\mathbb{E}_\infty$ spectra and* M *is a connective* R-S-*bimodule that is perfect as an* S-*module.*

Recall the following definition from [BGT13]:

Definition 4.4.2. A *split short exact sequence* (or semi-orthogonal decomposition) of stable ∞-categories is a diagram

$$\mathcal{C} \underset{\mathrm{L}_\mathcal{C}}{\overset{i_\mathcal{C}}{\rightleftarrows}} \mathcal{E} \underset{i_\mathcal{D}}{\overset{\mathrm{L}_\mathcal{D}}{\rightleftarrows}} \mathcal{D}$$

satisfying the following conditions:

(1) The functors $i_\mathcal{C}$ and $i_\mathcal{D}$ are fully faithful.

(2) The functor $\mathrm{L}_\mathcal{C}$ is right adjoint to $i_\mathcal{C}$, and $i_\mathcal{D}$ is right adjoint to $\mathrm{L}_\mathcal{D}$.

(3) The essential image of $i_\mathcal{D}$ is the right orthogonal to the essential image of $i_\mathcal{C}$.

Lemma 4.4.3. *Given a split short exact sequence of stable ∞-categories as above, the unit and counit maps of the adjunctions form an exact triangle*

$$i_\mathcal{C}\mathrm{L}_\mathcal{C}(\mathrm{X}) \to \mathrm{X} \to i_\mathcal{D}\mathrm{L}_\mathcal{D}(\mathrm{X}).$$

Proof. Condition (iii) of the definition implies that the fiber of the map $\mathrm{X} \to i_\mathcal{D}\mathrm{L}_\mathcal{D}\mathrm{X}$ has the form $i_\mathcal{C}\mathrm{Y}$ for some Y. Now $\mathrm{Y} \simeq \mathrm{L}_\mathcal{C} i_\mathcal{C}\mathrm{Y} \to \mathrm{L}_\mathcal{C}\mathrm{X}$ is an equivalence. □

The main result of this section is as follows:

Theorem 4.4.4. *Suppose given a split short exact sequence of stable ∞-categories:*

$$\mathcal{C} \underset{\mathrm{L}_\mathcal{C}}{\overset{i_\mathcal{C}}{\rightleftarrows}} \mathcal{E} \underset{i_\mathcal{D}}{\overset{\mathrm{L}_\mathcal{D}}{\rightleftarrows}} \mathcal{D}.$$

Assume that $\mathcal{C}$ and $\mathcal{D}$ admit weight structures such that $\pi_0\mathcal{E}(i_\mathcal{D}\mathrm{X}, i_\mathcal{C}\mathrm{Y}) = 0$ for any $\mathrm{X} \in \mathcal{D}_{w\leqslant 0}$ and $\mathrm{Y} \in \Sigma\mathcal{C}_{w\geqslant 0}$. Then there exists a unique weight structure on $\mathcal{E}$ such that all the functors in the diagram are weight-exact. Moreover, if $i_\mathcal{D}$ has a right adjoint $\mathrm{R}_\mathcal{D}$ and the weight structures on $\mathcal{C}$ and $\mathcal{D}$ admit adjacent t-structures, then so does the weight structure on $\mathcal{E}$.

The first part of the statement is related to [Bon10, 8.2]. However, it is not directly applicable here, as the gluing there requires the existence of two extra adjoints while the weight-exactness of all functors is not required.

To deduce Corollary 4.4.1 it suffices to set $\mathcal{C} = \mathbf{Perf}_{\mathrm{R}}, \mathcal{D} = \mathbf{Perf}_{\mathrm{S}}, \mathcal{E} = \mathbf{Perf}_{\mathrm{T}}$ with obvious functors between them. Note that the vanishing condition corresponds to connectivity of M while the existence of a right adjoint $\mathrm{R}_{\mathcal{D}}$ translates as perfectness of M.

4.4.5. *Proof of Theorem 4.4.4.*

Gluing the weight structure. We set $\mathcal{E}_{w\geqslant 0}$ and $\mathcal{E}_{w\leqslant 0}$ to be the extension-closures of the unions $i_{\mathcal{C}}\mathcal{C}_{w\geqslant 0} \cup i_{\mathcal{D}}\mathcal{D}_{w\geqslant 0}$ and $i_{\mathcal{C}}\mathcal{C}_{w\leqslant 0} \cup i_{\mathcal{D}}\mathcal{D}_{w\leqslant 0}$, respectively. These classes are semi-invariant with respect to translations, and the orthogonality axiom follows from fully faithfulness of $i_{\mathcal{C}}$ and $i_{\mathcal{D}}$ and from the extra condition in the statement. It suffices to construct a weight decomposition for an object $\mathrm{X} \in \mathcal{E}$.

By Lemma 4.4.3 we have an exact triangle $i_{\mathcal{C}}\mathrm{L}_{\mathcal{C}}\mathrm{X} \to \mathrm{X} \to i_{\mathcal{D}}\mathrm{L}_{\mathcal{D}}\mathrm{X}$. Take any weight decompositions of $\mathrm{L}_{\mathcal{C}}\mathrm{X}$ and $\mathrm{L}_{\mathcal{D}}\mathrm{X}$. Since there are no non-zero maps $\Omega i_{\mathcal{D}} w_{\leqslant 0}\mathrm{L}_{\mathcal{D}}\mathrm{X} \to i_{\mathcal{C}} w_{\geqslant 1}\mathrm{L}_{\mathcal{C}}\mathrm{X}$ we obtain a homotopy commutative square

$$\begin{array}{ccc} i_{\mathcal{D}} w_{\leqslant 0}\mathrm{L}_{\mathcal{D}}\Omega\mathrm{X} & \longrightarrow & i_{\mathcal{C}} w_{\leqslant 0}\mathrm{L}_{\mathcal{C}}\mathrm{X} \\ \downarrow & & \downarrow \\ i_{\mathcal{D}}\mathrm{L}_{\mathcal{D}}\Omega\mathrm{X} & \longrightarrow & i_{\mathcal{C}}\mathrm{Y} \end{array}$$

Using Lemma 1.1.11 of [BBD82] we get a commutative diagram

$$\begin{array}{ccccc} i_{\mathcal{C}} w_{\leqslant 0}\mathrm{L}_{\mathcal{C}}\mathrm{X} & \longrightarrow & \mathrm{X}' & \longrightarrow & i_{\mathcal{D}} w_{\leqslant 0}\mathrm{L}_{\mathcal{D}}\mathrm{X} \\ \downarrow & & \downarrow & & \downarrow \\ i_{\mathcal{C}}\mathrm{L}_{\mathcal{C}}\mathrm{X} & \longrightarrow & \mathrm{X} & \longrightarrow & i_{\mathcal{D}}\mathrm{L}_{\mathcal{D}}\mathrm{X} \\ \downarrow & & \downarrow & & \downarrow \\ i_{\mathcal{C}} w_{\geqslant 1}\mathrm{L}_{\mathcal{C}}\mathrm{X} & \longrightarrow & \mathrm{X}'' & \longrightarrow & i_{\mathcal{D}} w_{\geqslant 1}\mathrm{L}_{\mathcal{D}}\mathrm{X} \end{array}$$

in the homotopy category of $\mathcal{E}$, whose rows and columns come from exact triangles. In particular, $\mathrm{X}' \in \mathcal{E}_{w\leqslant 0}$ and $\mathrm{X}'' \in \Sigma\mathcal{E}_{w\geqslant 0}$, so the exact triangle $\mathrm{X}' \to \mathrm{X} \to \mathrm{X}''$ is a weight decomposition.

Gluing the adjacent t-structure. We now assume that the weight structures on $\mathcal{C}$ and $\mathcal{D}$ admit adjacent t-structures, and that i_{D} admits a right adjoint R_{D}. In this case we will show that the weight structure on $\mathcal{E}$ constructed above also admits an adjacent t-structure.

It suffices to construct maps $\tau\colon \tau_{\geqslant 0}\mathrm{X} \to \mathrm{X}$ for all X where $\tau_{\geqslant 0}\mathrm{X} \in \mathcal{E}_{w\geqslant 0}$ and

$$\pi_0\mathcal{E}(\mathrm{Y}, \mathrm{Cofib}(\tau)) = 0 \text{ for any } \mathrm{Y} \in \mathcal{E}_{w\geqslant 0}.$$

First consider the case $\mathrm{X} = i_{\mathcal{D}}\mathrm{Y}$. Then the map $i_{\mathcal{D}}\tau\colon i_{\mathcal{D}}\tau_{\geqslant 0}\mathrm{Y} \to i_{\mathcal{D}}\mathrm{Y}$ satisfies the conditions. Indeed,

$$\pi_0\mathcal{E}(i_{\mathcal{D}}\mathrm{Z}, \mathrm{Cofib}(i_{\mathcal{D}}\tau)) = \pi_0\mathcal{E}(i_{\mathcal{D}}\mathrm{Z}, i_{\mathcal{D}}\,\mathrm{Cofib}(\tau)) = \pi_0\mathcal{D}(\mathrm{Z}, \mathrm{Cofib}(\tau)) = 0$$

for any $\mathrm{Z} \in \mathcal{D}_{w\geqslant 0}$ and

$$\pi_0\mathcal{E}(i_{\mathcal{C}}\mathrm{Z}, \mathrm{Cofib}(i_{\mathcal{D}}\tau)) = \pi_0\mathcal{E}(i_{\mathcal{C}}\mathrm{Z}, i_{\mathcal{D}}\,\mathrm{Cofib}(\tau)) = \pi_0\mathcal{C}(\mathrm{Z}, \mathrm{L}_{\mathcal{C}} i_{\mathcal{D}}\,\mathrm{Cofib}(\tau)) = 0$$

for any $\mathrm{Z} \in \mathcal{C}$. So, $\pi_0\mathcal{E}(\mathrm{Z}, \mathrm{Cofib}(i_{\mathcal{D}}\tau)) = 0$ for any Z from the extension-closure of $i_{\mathcal{C}}\mathcal{C}_{w\geqslant 0} \cup i_{\mathcal{D}}\mathcal{D}_{w\geqslant 0}$.

Next consider the case $\mathrm{X} = i_{\mathcal{C}}\mathrm{Y}$. The cofiber Y' of the map $i_{\mathcal{C}}\tau\colon i_{\mathcal{C}}\tau_{\geqslant 0}\mathrm{Y} \to i_{\mathcal{C}}\mathrm{Y}$ satisfies

$$\pi_0\mathcal{E}(i_{\mathcal{C}}\mathrm{Z}, \mathrm{Y}') = \pi_0\mathcal{C}(\mathrm{Z}, \mathrm{Cofib}(\tau)) = 0.$$

for any $\mathrm{Z} \in \mathcal{C}_{w\geqslant 0}$. This orthogonality property is also satisfied by the cofiber Y'' of the map $i_{\mathcal{D}}\tau_{\geqslant 0}\mathrm{R}_{\mathcal{D}}\mathrm{Y}' \to i_{\mathcal{D}}\mathrm{R}_{\mathcal{D}}\mathrm{Y}' \to \mathrm{Y}'$ since

$$\pi_0\mathcal{E}(i_{\mathcal{C}}\mathrm{Z}, i_{\mathcal{D}}\tau_{\geqslant 0}\mathrm{R}_{\mathcal{D}}\mathrm{Y}') = \pi_0\mathcal{E}(\mathrm{Z}, \mathrm{L}_{\mathcal{C}} i_{\mathcal{D}}\tau_{\geqslant 0}\mathrm{R}_{\mathcal{D}}\mathrm{Y}') = 0$$

for any $Z \in \mathcal{C}$. The map

$$\pi_0\mathcal{E}(i_{\mathcal{D}}Z, i_{\mathcal{D}}\tau_{\geqslant 0}R_{\mathcal{D}}Y') \simeq \pi_0\mathcal{D}(Z, \tau_{\geqslant 0}R_{\mathcal{D}}Y') \to \pi_0\mathcal{D}(Z, R_{\mathcal{D}}Y') \simeq \pi_0\mathcal{E}(i_{\mathcal{D}}Z, Y')$$

is an isomorphism for $Z \in \mathcal{D}_{w\geqslant 0}$ and an injection for $Z \in \Omega\mathcal{D}_{w\geqslant 0}$ by orthogonality properties of the t-structure on $\mathcal{D}$. Therefore from the long exact sequence associated to an exact triangle we see that $\pi_0\mathcal{E}(i_{\mathcal{D}}Z, Y'') = 0$ for any $Z \in \mathcal{D}_{w\geqslant 0}$. The fiber F of the map $X \to Y''$ is an extension of $i_{\mathcal{D}}\tau_{\geqslant 0}R_{\mathcal{D}}Y' = \mathrm{Fib}(Y' \to Y'')$ by $i_{\mathcal{C}}\tau_{\geqslant 0}Y = \mathrm{Fib}(X \to Y')$, so it belongs to $\mathcal{E}_{w\geqslant 0}$ and the map $F \to X$ satisfies the desired conditions.

Finally let X be arbitrary. By Lemma 4.4.3 there is an exact triangle $\Omega i_{\mathcal{D}}L_{\mathcal{D}}X \xrightarrow{p} i_{\mathcal{C}}L_{\mathcal{C}}X \to X$. The already constructed maps $\tau_{\geqslant 0}\Omega i_{\mathcal{D}}L_{\mathcal{D}} \to \Omega i_{\mathcal{D}}L_{\mathcal{D}}X$ and $\tau_{\geqslant 0}i_{\mathcal{C}}L_{\mathcal{C}}X \to i_{\mathcal{C}}L_{\mathcal{C}}X$ fit into a homotopy commutative diagram

$$\begin{array}{ccc} \tau_{\geqslant 0}\Omega i_{\mathcal{D}}L_{\mathcal{D}}X & \overset{\tau_{\geqslant 0}p}{\dashrightarrow} & \tau_{\geqslant 0}i_{\mathcal{C}}L_{\mathcal{C}}X \\ \downarrow{\scriptstyle \tau_{\mathcal{D}}} & & \downarrow{\scriptstyle \tau_{\mathcal{C}}} \\ \Omega i_{\mathcal{D}}L_{\mathcal{D}}X & \xrightarrow{p} & i_{\mathcal{C}}L_{\mathcal{C}}X. \end{array}$$

Applying Lemma 1.1.11 of [BBD82] we get a homotopy commutative diagram

$$\begin{array}{ccccc} \tau_{\geqslant 0}\Omega i_{\mathcal{D}}L_{\mathcal{D}}X & \xrightarrow{\tau_{\geqslant 0}p} & \tau_{\geqslant 0}i_{\mathcal{C}}L_{\mathcal{C}}X & \longrightarrow & P \\ \downarrow{\scriptstyle \tau_{\mathcal{D}}} & & \downarrow{\scriptstyle \tau_{\mathcal{C}}} & & \downarrow \\ \Omega i_{\mathcal{D}}L_{\mathcal{D}}X & \xrightarrow{p} & i_{\mathcal{C}}L_{\mathcal{C}}X & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ \tau_{\leqslant -1}\Omega i_{\mathcal{D}}L_{\mathcal{D}}X & \longrightarrow & \tau_{\leqslant -1}i_{\mathcal{C}}L_{\mathcal{C}}X & \longrightarrow & N \end{array}$$

whose rows and columns are exact triangles. We note that P belongs to $\mathcal{E}_{w\geqslant 0}$ and $\pi_0\mathcal{E}(Y, N) = 0$ for any $Y \in \mathcal{E}_{w\geqslant 1}$. Moreover, $\tau_{\leqslant -1}\Omega i_{\mathcal{D}}L_{\mathcal{D}}X = i_{\mathcal{D}}\tau_{\leqslant -1}\Omega L_{\mathcal{D}}X$ by construction of $\tau_{\mathcal{D}}$, so $\pi_0\mathcal{E}(Y, N) = 0$ for any $Y \in i_{\mathcal{C}}\mathcal{C}_{w\geqslant 0}$. However, the group $\pi_0\mathcal{E}(i_{\mathcal{D}}Y, N)$ might be non-zero for $Y \in \mathcal{D}^{\heartsuit}$. The issue is fixed using the following.

Claim. There exists an object $I \in \mathrm{Ht}_{\mathcal{D}}$ and a map $\tau_0 \colon i_{\mathcal{D}}I \to N$ such that the map $\pi_0\mathcal{E}(i_{\mathcal{D}}Y, \tau_0)$ is an isomorphism for any $Y \in \mathcal{D}^{\heartsuit}$.

Indeed, assume the claim is proven. Then it implies the vanishing of $\pi_0\mathcal{E}(i_{\mathcal{D}}Y, \mathrm{Cofib}(\tau_0))$ for any $Y \in \mathcal{D}_{w\geqslant 0}$. Moreover, $\pi_0\mathcal{E}(i_{\mathcal{C}}Y, \mathrm{Cofib}(\tau_0))$ is just isomorphic to $\pi_0\mathcal{E}(i_{\mathcal{C}}Y, N)$ for $Y \in \mathcal{C}$, so $\pi_0(Y, \mathrm{Cofib}(\tau_0)) = 0$ for any $Y \in \mathcal{E}_{w\geqslant 0}$. The fiber $\tau_{\geqslant 0}X$ of the map $X \to \mathrm{Cofib}(\tau_0)$ belongs to $\mathcal{E}_{w\geqslant 0}$ as it is an extension of $i_{\mathcal{D}}I$ by P. So $\tau_{\geqslant 0}X \to X$ gives us the desired map. It suffices to prove the claim.

From the bottom and the middle exact triangles in the diagram above we see that for $Y \in \mathcal{D}^{\heartsuit}$

$$\pi_0\mathcal{E}(i_{\mathcal{D}}Y, N) \simeq \mathrm{Im}(\pi_0\mathcal{E}(i_{\mathcal{D}}Y, X) \xrightarrow{u} \pi_0\mathcal{E}(i_{\mathcal{D}}Y, \Sigma\tau_{\leqslant -1}\Omega i_{\mathcal{D}}L_{\mathcal{D}}X))$$

and the isomorphism is induced by the map $N \to \Sigma\tau_{\leqslant -1}\Omega i_{\mathcal{D}}L_{\mathcal{D}}X$.

Since $i_{\mathcal{D}}$ is fully faithful, the composition of the unit and the counit $i_{\mathcal{D}}R_{\mathcal{D}}X \to X \to i_{\mathcal{D}}L_{\mathcal{D}}X$ defines a map $\gamma \colon R_{\mathcal{D}}X \to L_{\mathcal{D}}X$. We define I to be the image in the abelian category $\mathrm{Ht}_{\mathcal{D}}$ of the map $\pi_0\gamma \colon \pi_0R_{\mathcal{D}}X \to \pi_0L_{\mathcal{D}}X$. There is an obvious map f from $i_{\mathcal{D}}I$ to $\Sigma\tau_{\leqslant -1}\Omega i_{\mathcal{D}}L_{\mathcal{D}}X = i_{\mathcal{D}}\tau_{\leqslant 0}L_{\mathcal{D}}X$. The following commutative diagram shows that f maps $\pi_0\mathcal{E}(i_{\mathcal{D}}Y, i_{\mathcal{D}}I)$ isomorphically to the subset of $\pi_0\mathcal{E}(i_{\mathcal{D}}Y, i_{\mathcal{D}}\tau_{\leqslant 0}L_{\mathcal{D}}X)$ isomorphic to $\pi_0\mathcal{E}(i_{\mathcal{D}}Y, N)$.

$$\begin{array}{ccccc}
\pi_0\mathcal{E}(i_{\mathcal{D}}\mathrm{Y}, i_{\mathcal{D}}\pi_0\mathrm{R}_{\mathcal{D}}\mathrm{X}) & \twoheadrightarrow & \pi_0\mathcal{E}(i_{\mathcal{D}}\mathrm{Y}, i_{\mathcal{D}}\mathrm{I}) & \hookrightarrow & \pi_0\mathcal{E}(i_{\mathcal{D}}\mathrm{Y}, i_{\mathcal{D}}\pi_0\mathrm{L}_{\mathcal{D}}\mathrm{X}) \\
\downarrow{\scriptstyle (1)} & & & \searrow^{f} & \downarrow{\scriptstyle (2)} \\
\pi_0\mathcal{E}(i_{\mathcal{D}}\mathrm{Y}, i_{\mathcal{D}}\tau_{\leqslant 0}\mathrm{R}_{\mathcal{D}}\mathrm{X}) & & \longrightarrow & & \pi_0\mathcal{E}(i_{\mathcal{D}}\mathrm{Y}, i_{\mathcal{D}}\tau_{\leqslant 0}\mathrm{L}_{\mathcal{D}}\mathrm{X}) \\
\uparrow{\scriptstyle (3)} & & & \nearrow^{u} & =\uparrow \\
\pi_0\mathcal{E}(i_{\mathcal{D}}\mathrm{Y}, i_{\mathcal{D}}\mathrm{R}_{\mathcal{D}}\mathrm{X}) & \xrightarrow{(4)} & \pi_0\mathcal{E}(i_{\mathcal{D}}\mathrm{Y}, \mathrm{X}) & \longrightarrow & \pi_0\mathcal{E}(i_{\mathcal{D}}\mathrm{Y}, i_{\mathcal{D}}\tau_{\leqslant 0}\mathrm{L}_{\mathcal{D}}\mathrm{X})
\end{array}$$

Indeed, considering the long exact sequences the maps (1) and (2) fit into, we see that they are isomorphisms by the orthogonality axiom for weight structures. The map (3) is a surjection for the same reason. The map (4) becomes the map $\pi_0\mathcal{D}(\mathrm{Y}, \mathrm{R}_{\mathcal{D}}i_{\mathcal{D}}\mathrm{R}_{\mathcal{D}}\mathrm{X}) \to \pi_0\mathcal{D}(\mathrm{Y}, \mathrm{R}_{\mathcal{D}}\mathrm{X})$ under the adjunction isomorphism for $i_{\mathcal{D}}$ and $\mathrm{R}_{\mathcal{D}}$. The latter map is an isomorphism because $\mathrm{id}_{\mathcal{D}} \to \mathrm{R}_{\mathcal{D}}i_{\mathcal{D}}$ is an equivalence. Therefore we see that the images of $\pi_0\mathcal{E}(i_{\mathcal{D}}\mathrm{Y}, i_{\mathcal{D}}\mathrm{I})$ and of $\pi_0\mathcal{E}(i_{\mathcal{D}}\mathrm{Y}, \mathrm{X})$ in $\pi_0\mathcal{E}(i_{\mathcal{D}}\mathrm{Y}, i_{\mathcal{D}}\tau_{\leqslant 0}\mathrm{L}_{\mathcal{D}}\mathrm{X})$ coincide and the former maps injectively onto the image.

Now it suffices to construct a map $i_{\mathcal{D}}\mathrm{I} \to \mathrm{N}$ that makes the triangle

$$\begin{array}{ccc}
\mathrm{N} & \longrightarrow & i_{\mathcal{D}}\tau_{\leqslant 0}\mathrm{L}_{\mathcal{D}}\mathrm{X} \\
\uparrow & \nearrow^{f} & \\
i_{\mathcal{D}}\mathrm{I} & &
\end{array}$$

commute. Since N is a fiber of the map $i_{\mathcal{D}}\tau_{\leqslant 0}\mathrm{L}_{\mathcal{D}}\mathrm{X} \to \Sigma\tau_{\leqslant -1}i_{\mathcal{C}}\mathrm{L}_{\mathcal{C}}\mathrm{X}$, it suffices to prove that the composite map $i_{\mathcal{D}}\mathrm{I} \to \Sigma\tau_{\leqslant -1}i_{\mathcal{C}}\mathrm{L}_{\mathcal{C}}\mathrm{X}$ is null-homotopic. We will show that the element f in the upper left corner of the following commutative diagram becomes trivial after applying the horizontal map.

$$\begin{array}{ccc}
\pi_0\mathcal{E}(i_{\mathcal{D}}\mathrm{I}, i_{\mathcal{D}}\tau_{\leqslant 0}\mathrm{L}_{\mathcal{D}}\mathrm{X}) & \longrightarrow & \pi_0\mathcal{E}(i_{\mathcal{D}}\mathrm{I}, \Sigma\tau_{\leqslant -1}i_{\mathcal{C}}\mathrm{L}_{\mathcal{C}}\mathrm{X}) \\
\downarrow{\scriptstyle (1)} & & \downarrow{\scriptstyle (2)} \\
\pi_0\mathcal{E}(i_{\mathcal{D}}\pi_0\mathrm{R}_{\mathcal{D}}\mathrm{X}, i_{\mathcal{D}}\tau_{\leqslant 0}\mathrm{L}_{\mathcal{D}}\mathrm{X}) & \longrightarrow & \pi_0\mathcal{E}(i_{\mathcal{D}}\pi_0\mathrm{R}_{\mathcal{D}}\mathrm{X}, \Sigma\tau_{\leqslant -1}i_{\mathcal{C}}\mathrm{L}_{\mathcal{C}}\mathrm{X}) \\
\uparrow{\scriptstyle (3)} & & \uparrow \\
\pi_0\mathcal{E}(i_{\mathcal{D}}\tau_{\leqslant 0}\mathrm{R}_{\mathcal{D}}\mathrm{X}, i_{\mathcal{D}}\tau_{\leqslant 0}\mathrm{L}_{\mathcal{D}}\mathrm{X}) & \longrightarrow & \pi_0\mathcal{E}(i_{\mathcal{D}}\tau_{\leqslant 0}\mathrm{R}_{\mathcal{D}}\mathrm{X}, \Sigma\tau_{\leqslant -1}i_{\mathcal{C}}\mathrm{L}_{\mathcal{C}}\mathrm{X}) \\
\downarrow{\scriptstyle (4)} & & \downarrow{\scriptstyle (5)} \\
\pi_0\mathcal{E}(i_{\mathcal{D}}\mathrm{R}_{\mathcal{D}}\mathrm{X}, i_{\mathcal{D}}\tau_{\leqslant 0}\mathrm{L}_{\mathcal{D}}\mathrm{X}) & \longrightarrow & \pi_0\mathcal{E}(i_{\mathcal{D}}\mathrm{R}_{\mathcal{D}}\mathrm{X}, \Sigma\tau_{\leqslant -1}i_{\mathcal{C}}\mathrm{L}_{\mathcal{C}}\mathrm{X})
\end{array}$$

The long exact sequence associated to the exact triangle

$$i_{\mathcal{D}}\mathrm{K} \to i_{\mathcal{D}}\pi_0\mathrm{R}_{\mathcal{D}}\mathrm{X} \to i_{\mathcal{D}}\mathrm{I}$$

where K denotes the kernel of the surjection $\pi_0\mathrm{R}_{\mathcal{D}}\mathrm{X} \to \mathrm{I}$ in the abelian category $\mathrm{Ht}_{\mathcal{D}}$, together with the vanishing property in the definition of $\tau_{\mathcal{C}}$ and $\tau_{\mathcal{D}}$, yields injectivity of the maps (1) and (2). So it suffices to show that the image of f in $\pi_0\mathcal{E}(i_{\mathcal{D}}\pi_0\mathrm{R}_{\mathcal{D}}\mathrm{X}, i_{\mathcal{D}}\tau_{\leqslant 0}\mathrm{L}_{\mathcal{D}}\mathrm{X})$ is mapped to zero via the horizontal map. By definition of f the image lifts via (3) to a map $\tilde{f} = i_{\mathcal{D}}\tau_{\leqslant 0}(\gamma)$. The long exact sequence associated with the exact triangle

$$i_{\mathcal{D}}\tau_{\geqslant 1}\mathrm{R}_{\mathcal{D}}\mathrm{X} \to i_{\mathcal{D}}\mathrm{R}_{\mathcal{D}}\mathrm{X} \to i_{\mathcal{D}}\tau_{\leqslant 0}\mathrm{R}_{\mathcal{D}}\mathrm{X}$$

together with the vanishing property in the definition of $\tau_{\mathcal{C}}$ and $\tau_{\mathcal{D}}$ implies that the maps (4) and (5) are isomorphisms. So it suffices to show that the image of $\tilde{f}$ via (4) is mapped to zero via the horizontal map. The image of $\tilde{f}$ is the composite $i_{\mathcal{D}}\mathrm{R}_{\mathcal{D}}\mathrm{X} \to i_{\mathcal{D}}\tau_{\leqslant 0}\mathrm{R}_{\mathcal{D}}\mathrm{X} \to i_{\mathcal{D}}\tau_{\leqslant 0}\mathrm{L}_{\mathcal{D}}\mathrm{X}$. This factorizes as $i_{\mathcal{D}}\mathrm{R}_{\mathcal{D}}\mathrm{X} \to i_{\mathcal{D}}\mathrm{L}_{\mathcal{D}}\mathrm{X} \to i_{\mathcal{D}}\tau_{\leqslant 0}\mathrm{L}_{\mathcal{D}}\mathrm{X}$. Therefore, the result follows from the fact that the composite of the two maps in an exact triangle is null-homotopic.

5. Nilpotent extensions via weights

In this section, we study a notion of nilpotent extensions of stable ∞-categories endowed with a bounded weight structure. This will be a generalization of a nilpotent extension of connective $\mathbb{E}_1$-rings. We recall that the latter means an $\mathbb{E}_1$-morphism of connective $\mathbb{E}_1$-rings $\mathrm{R} \to \mathrm{S}$ such that the induced map $\pi_0(\mathrm{R}) \to \pi_0(\mathrm{S})$ has nilpotent kernel. We achieve that by first working out this notion and its properties for additive ∞-categories, and then passing to the world of stable ∞-categories via Theorem 3.2.7.

5.1. Nilpotent extensions of additive ∞-categories.

Definition 5.1.1. A **nilpotent extension** of additive ∞-categories is an additive functor between additive ∞-categories:

$$f : \mathcal{A} \to \mathcal{B},$$

such that:

(1) f is essentially surjective,
(2) for all objects $x, y \in \mathcal{A}$ the map

$$\pi_0 \operatorname{Maps}_{\mathcal{A}}(x, y) \to \pi_0 \operatorname{Maps}_{\mathcal{B}}(f(x), f(y))$$

is a surjection,
(3) there exists $n \in \mathbb{N}$ such that for any sequence of composable morphisms $f_1, \cdots, f_n \in \mathcal{A}$ for which each $f(f_i)$ is trivial[7], the composition $f_1 \circ \cdots \circ f_n$ is trivial.

The third condition is equivalent to saying that the kernel ideal of the functor $h(\mathcal{A}) \to h(\mathcal{B})$ is nilpotent.

Remark 5.1.2. We note that the integer n that appears in part (3) of Definition 5.1.1 only depends on the additive functor f.

Proposition 5.1.3. *Suppose that $f : \mathcal{A} \to \mathcal{B}$ is a nilpotent extension of additive ∞-categories. Then f is conservative.*

Proof. Let $g : x \to y$ be a morphism in $\mathcal{A}$. By hypothesis, $f(g)$ is invertible in $\mathcal{B}$, hence we may pick an inverse $h' : f(y) \to f(x)$. Using (2) of Definition 5.1.1, we may choose a lift of h' which we call h. We claim that h is an inverse of g. Indeed, it suffices to prove that gh and hg are invertible. Since the arguments are the same we do it for gh. First note that $f(\mathrm{id} - gh) \simeq \mathrm{id} - f(g)f(h) \simeq 0$ since f is an additive functor and $f(h)$ is an inverse of $f(g)$. Therefore, by assumption (3) of Definition 5.1.1, there exists an n such that $(\mathrm{id} - gh)^n \simeq 0$. But now, we can furnish and inverse to gh given by

$$(gh)^{-1} \simeq (\mathrm{id} - (\mathrm{id} - gh))^{-1} \simeq \mathrm{id} + (\mathrm{id} - gh) + (\mathrm{id} - gh)(\mathrm{id} - gh) + \cdots + (\mathrm{id} - gh)^{n-1}.$$

□

Remark 5.1.4. After Proposition 5.1.3, we can replace part (1) of Definition 5.1.1 by saying that the induced map on equivalence classes of objects:

$$\pi_0(\mathcal{A}^{\simeq}) \to \pi_0(\mathcal{B}^{\simeq})$$

is a bijection.

[7]We note that any additive ∞-category is pointed, hence a map $f : x \to y$ is said to be **trivial** if it is homotopic to $f : x \to 0 \to y$.

Example 5.1.5. Suppose that R, S are connective $\mathbb{E}_1$-ring spectra. Then we can consider the additive ∞-categories $\mathbf{Free}_{\mathrm{R}}$, $\mathbf{Free}_{\mathrm{S}}$ of free modules over these rings. A morphism of $\mathbb{E}_1$-rings $f : \mathrm{R} \to \mathrm{S}$ then defines an additive functor $f : \mathbf{Free}_{\mathrm{R}} \to \mathbf{Free}_{\mathrm{S}}$. The requirement that the extension is nilpotent is then equivalent to asking that the map $\pi_0(\mathrm{R}) \to \pi_0(\mathrm{S})$ has a kernel which is a nilpotent ideal.

Example 5.1.6. What follows is arguably the most important example from the point of view of this paper in the same way that if A is a “derived ring" (an animated ring or a connective $\mathbb{E}_1$-algebra) then $\mathrm{A} \to \pi_0(\mathrm{A})$ is a nilpotent extension. Let $\mathcal{A}$ be an additive ∞-category and let $h(\mathcal{A})$ be its homotopy category which is a 1-category. Then the map $\mathcal{A} \to h(\mathcal{A})$ is evidently a nilpotent extension of additive ∞-categories.

5.1.7. *Square zero extensions of additive ∞-categories.* In this next section, we furnish a large collection of examples of nilpotent extensions of additive ∞-categories by introducing the notion of a square zero extension of additive ∞-categories. We emphasize that, unlike the proof of the DGM theorem for connective $\mathbb{E}_1$-ring spectra, this theory is *not* necessary for the proof. On the other hand, they provide a large collection of new functors for which we can prove the DGM theorem.

Definition 5.1.8. Let $\mathcal{A}$ be an additive ∞-category. An **$\mathcal{A}$-bimodule** is a biadditive[8] functor

$$\mathcal{M} : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathrm{Spt}_{\geqslant 0}.$$

By adjunction the functor gives rise to an additive functor

$$\mathcal{M} : \mathcal{A} \to \mathrm{Fun}^{\times}(\mathcal{A}^{\mathrm{op}}, \mathrm{Spt})$$

which we refer to by the same letter.

Remark 5.1.9. Unpacking the structure of an $\mathcal{A}$-bimodule, we get the following pieces of data:

(1) Fixing an object $y \in \mathcal{A}$, for any morphism $g : c \to c'$ in $\mathcal{A}$ we get a map of connective spectra:
$$g^* : \mathcal{M}(c', y) \to \mathcal{M}(c, y),$$
as part of the functor
$$\mathcal{M}(-, y) : \mathcal{A}^{\mathrm{op}} \to \mathrm{Spt}_{\geqslant 0}.$$
In particular, this endows $\mathcal{M}(y, y) \in \mathrm{Spt}_{\geqslant 0}$ with the structure of a right $\mathbf{End}_{\mathcal{A}}(y, y)$-module.

(2) Fixing an object $z \in \mathcal{A}$, for any morphism $g : c \to c'$ in $\mathcal{A}$ we get a map of connective spectra:
$$g_* : \mathcal{M}(z, c) \to \mathcal{M}(z, c'),$$
as part of the functor
$$\mathcal{M}(z, -) : \mathcal{A} \to \mathrm{Spt}_{\geqslant 0}.$$
In particular, this endows $\mathcal{M}(z, z) \in \mathrm{Spt}_{\geqslant 0}$ the structure of a left $\mathbf{End}_{\mathcal{A}}(z, z)$-module.

(3) Altogether, we can package the two structures above by saying that for each $z \in \mathcal{C}$, $\mathcal{M}(z, z)$ is an $\mathbf{End}_{\mathcal{A}}(z, z)$-bimodule.

We would like to make sense of the square zero extension of additive ∞-categories, which we denote as $\mathcal{A} \oplus \mathcal{M}$. To do so, we first recall the formalism of lax equalizers [NS18, Section II.1] which is a special case of lax pullbacks as in [Tam18, Section I].

[8] By this we mean a functor which is additive in each variable. In the presence of tensor products of additive ∞-categories as in [Lur18, Sections 10.1.6, D.2.1.1], we can say that such a functor defines an additive functor $\mathcal{A}^{\mathrm{op}} \otimes \mathcal{A} \to \mathrm{Spt}_{\geqslant 0}$

Construction 5.1.10. Let $\mathcal{A}, \mathcal{B}$ be ∞-categories equipped with functors $\mathrm{F}, \mathrm{G} : \mathcal{A} \to \mathcal{B}$. The **lax equalizer** of F and G

$$\mathrm{LEq}(\mathrm{F}, \mathrm{G})$$

is the ∞-category given by the pullback

$$\begin{array}{ccc} \mathrm{LEq}(\mathrm{F},\mathrm{G}) & \longrightarrow & \mathrm{Fun}(\Delta^1, \mathcal{B}) \\ \downarrow & & \downarrow^{(\mathrm{res}_{\delta_0}, \mathrm{res}_{\delta_1})} \\ \mathcal{A} & \xrightarrow{(\mathrm{F},\mathrm{G})} & \mathcal{B} \times \mathcal{B}. \end{array}$$

The objects of $\mathrm{LEq}(\mathrm{F}, \mathrm{G})$ are given by pairs (c, f) where $c \in \mathcal{A}$ and a morphism $f : \mathrm{F}(c) \to \mathrm{G}(c)$ in $\mathcal{B}$. The mapping spaces in $\mathrm{LEq}(\mathrm{F}, \mathrm{G})$ are given by an equalizer diagram

$$\mathrm{Maps}_{\mathrm{LEq}(\mathrm{F},\mathrm{G})}((c, f), (c', f'))) \to \mathrm{Maps}_{\mathcal{A}}(c, c') \rightrightarrows \mathrm{Maps}(\mathrm{F}(c), \mathrm{G}(c'))$$

where one of the right maps is given by

$$\mathrm{Maps}_{\mathcal{A}}(c, c') \to \mathrm{Maps}_{\mathcal{B}}(\mathrm{F}c, \mathrm{F}c') \xrightarrow{f'_*} \mathrm{Maps}_{\mathcal{B}}(\mathrm{F}c, \mathrm{G}c'),$$

and the other is

$$\mathrm{Maps}_{\mathcal{A}}(c, c') \to \mathrm{Maps}_{\mathcal{B}}(\mathrm{G}c, \mathrm{G}c') \xrightarrow{f^*} \mathrm{Maps}_{\mathcal{B}}(\mathrm{F}c, \mathrm{G}c').$$

Lemma 5.1.11. *Suppose that $\mathcal{A}, \mathcal{B}$ are additive ∞-categories with additive functors*

$$\mathrm{F}, \mathrm{G} : \mathcal{A} \to \mathcal{B}.$$

Then, $\mathrm{LEq}(\mathrm{F}, \mathrm{G})$ *is an additive ∞-category and the functor* $\mathrm{LEq}(\mathrm{F}, \mathrm{G}) \to \mathcal{A}$ *is additive.*

Proof. $\mathrm{Fun}(\Delta^1, \mathcal{B})$ is additive since limits and colimits in a functor category are computed pointwise (see [Lur17b, Corollary 5.1.2.3]). It suffices to show that a pullback of a diagram of additive ∞-categories is also an additive ∞-category. This follows from general considerations about pullbacks of ∞-categories: if we have a cospan $\mathcal{A}_0 \xrightarrow{f} \mathcal{A}_1 \xleftarrow{g} \mathcal{A}_2$, then K-shaped limits and colimits in the pullback ∞-category are computed pointwise if these limits and colimits are preserved by f and g. This already proves that the pullback category is preadditive. To check the additivity condition we need only prove that the shear map is an equivalence, but equivalences in the pullback category are tested pointwise. □

Construction 5.1.12. Given an additive ∞-category $\mathcal{A}$ and an $\mathcal{A}$-bimodule $\mathcal{M}$, we construct the **square zero extension** $\mathcal{A} \oplus \mathcal{M}$ in the following manner: take the (pointwise) suspension of $\mathcal{M}$ and adjoint to get a functor

$$\mathcal{A} \xrightarrow{\Sigma\mathcal{M}} \mathrm{Fun}^{\times}(\mathcal{A}^{\mathrm{op}}, \mathrm{Spt}).$$

The additive ∞-category $\mathcal{A} \oplus \mathcal{M}$ is then defined as the lax equalizer between the Yoneda functor $\mathcal{A} \xrightarrow{よ} \mathrm{Fun}^{\times}(\mathcal{A}^{\mathrm{op}}, \mathrm{Spt})$ and $\Sigma\mathcal{M}$:

$$\mathcal{A} \oplus \mathcal{M} := \mathrm{LEq}(よ, \Sigma\mathcal{M}).$$

In other words, it is the pullback

$$\begin{array}{ccc} \mathcal{A} \oplus \mathcal{M} & \longrightarrow & \mathrm{Fun}(\Delta^1, \mathrm{Fun}^{\times}(\mathcal{A}^{\mathrm{op}}, \mathrm{Spt})) \\ {}^{p}\downarrow & & \downarrow^{(\mathrm{res}_{\delta_0}, \mathrm{res}_{\delta_1})} \\ \mathcal{A} & \xrightarrow{(よ, \Sigma\mathcal{M})} & \mathrm{Fun}^{\times}(\mathcal{A}^{\mathrm{op}}, \mathrm{Spt}) \times \mathrm{Fun}^{\times}(\mathcal{A}^{\mathrm{op}}, \mathrm{Spt}). \end{array}$$

Lemma 5.1.13. *The ∞-category $\mathcal{A} \oplus \mathcal{M}$ is an additive ∞-category and the functor $p : \mathcal{A} \oplus \mathcal{M} \to \mathcal{A}$ is additive.*

Proof. This follows from Lemma 5.1.11. □

Remark 5.1.14. Let us unpack the ∞-category $\mathcal{A} \oplus \mathcal{M}$. Its objects are pairs

$$x = (c \in \mathcal{A}, t : \text{よ}(c) \to \Sigma\mathcal{M}(-, c)).$$

Now, since $\mathcal{M}$ takes values in connective spectra, we have that $\pi_0(\Sigma\mathcal{M}(c, c)) = 0$, hence the data of t is, in a sense made precise in Lemma 5.1.19, redundant. Moreover, we can compute the mapping (connective) spectrum

$$\mathbf{Maps}_{\mathcal{A}\oplus\mathcal{M}}((c, t), (c', t'))$$

as the equalizer of the maps (notation as in Remark 5.1.9)

$$\mathbf{Maps}_{\mathcal{A}}(c, c') \underset{f \mapsto f^* t}{\overset{f \mapsto f_* t'}{\rightrightarrows}} \Sigma\mathcal{M}(c, c'). \tag{5.1.15}$$

This means that a morphism $x = (c, t) \to x' = (c', t')$ can be viewed as the data of a morphism $f : c \to c'$ in $\mathcal{A}$ and a "loop" in $\Sigma\mathcal{M}(c, c')$ identifying f^*t with f_*t', i.e., a point in $\mathcal{M}(c, c')$.

The fiber sequence defining the equalizer also gives rise to a fiber sequence

$$\mathcal{M}(p(x), p(x')) \overset{j_{x,x'}}{\to} \mathbf{Maps}_{\mathcal{A}\oplus\mathcal{M}}(x, x') \overset{p_{x,x'}}{\to} \mathbf{Maps}_{\mathcal{A}}(p(x), p(x')). \tag{5.1.16}$$

of functors landing in connective spectra

$$(\mathcal{A} \oplus \mathcal{M})^{\mathrm{op}} \times (\mathcal{A} \oplus \mathcal{M}) \to \mathrm{Spt}_{\geqslant 0}.$$

Lemma 5.1.17. *The map $\mathcal{A} \oplus \mathcal{M} \to \mathcal{A}$ admits a section*

$$i : \mathcal{A} \to \mathcal{A} \oplus \mathcal{M}$$

that sends an object c to $(c, 0)$.

Proof. The right vertical map in the diagram (5.1.10) admits a section given by the functor $(b, b') \mapsto (b \overset{0}{\to} b')$. Formally, this is the direct sum of functors

$$\mathcal{B} \times \mathcal{B} \overset{p_1}{\to} \mathcal{B} = \mathrm{Fun}(\Delta^0, \mathcal{B}) \overset{\mathrm{Ran}_{\delta_0}}{\to} \mathrm{Fun}(\Delta^1, \mathcal{B}) \text{ and}$$
$$\mathcal{B} \times \mathcal{B} \overset{p_2}{\to} \mathcal{B} = \mathrm{Fun}(\Delta^0, \mathcal{B}) \overset{\mathrm{Lan}_{\delta_1}}{\to} \mathrm{Fun}(\Delta^1, \mathcal{B}),$$

where Ran (resp. Lan) is right Kan extension (resp. Left Kan extension). This induces a section of the left vertical map by functoriality of pullbacks. It sends c to $(c, 0)$ by construction. ☐

Proposition 5.1.18. *The functor p defined above is a nilpotent extension of additive ∞-categories.*

Proof. Part (1) of Definition 5.1.1 follows from Lemma 5.1.17. Observing the fiber sequence (5.1.16) and using the fact that $\pi_0\Sigma\mathcal{M}(c, c') = 0$ we also see that Part (2) is satisfied. Let us prove (3); we claim that the n appearing in (3) is just 2. Indeed, suppose that $f : x \to x', g : x' \to x''$ are composable morphisms in $\mathcal{A} \oplus \mathcal{M}$ such that $p(f) \simeq p(g) \simeq 0$. We claim that $g \circ f \simeq 0$. Indeed, by the bifunctoriality of the fiber sequence (5.1.16) we have the following commutative diagram where the rows are fiber sequences of spectra, all of which are connective:

$$\begin{array}{ccccc}
\mathcal{M}(p(x'), p(x'')) & \xrightarrow{j_{x',x''}} & \mathbf{Maps}_{\mathcal{A}\oplus\mathcal{M}}(x', x'') & \longrightarrow & \mathbf{Maps}_{\mathcal{A}}(p(x'), p(x'')) \\
{\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\
\mathcal{M}(p(x), p(x'')) & \xrightarrow{j_{x,x''}} & \mathbf{Maps}_{\mathcal{A}\oplus\mathcal{M}}(x, x'') & \longrightarrow & \mathbf{Maps}_{\mathcal{A}}(p(x), p(x'')).
\end{array}$$

Now since $g \in \mathbf{Maps}_{\mathcal{A}\oplus\mathcal{M}}(x', x'')$ is such that $p(g) \simeq 0$, we can find $m \in \mathcal{M}(p(x'), p(x''))$ and an equivalence $j_{x',x''}(m) \simeq g$. On the other hand, since $p(f) \simeq 0$, we get that the left vertical f^* is nullhomotopic. Therefore, $g \circ f \simeq f^*(g) \simeq j_{x,x''} \circ f^*(m) \simeq 0$. ☐

Lemma 5.1.19. *Consider the subcategory $\mathcal{A} \oplus \mathcal{M}^0 \subset \mathcal{A} \oplus \mathcal{M}$ spanned by objects of the form $(c \in \mathcal{A}, 0)$. Then the inclusion functor $(\mathcal{A} \oplus \mathcal{M})^0 \hookrightarrow \mathcal{A} \oplus \mathcal{M}$ is an equivalence of categories.*

Proof. The functor in question is fully faithful, so it suffices to show essential surjectivity, i.e. that any object of $\mathcal{A} \oplus \mathcal{M}$ is equivalent to an object of the form $(c, 0)$. By Proposition 5.1.18 $\mathcal{A} \oplus \mathcal{M} \stackrel{p}{\to} \mathcal{A}$ is a nilpotent extension, in particular, it is surjective on homotopy classes of morphisms. Hence we can find a map $(c,t) \stackrel{f}{\to} (c,0)$ such that $p(f) \simeq \mathrm{id}_c$ for any object $(c,t) \in \mathcal{A} \oplus \mathcal{M}$. By Proposition 5.1.3 p is conservative, so f is an equivalence. □

The next proposition gives us a way to compute composition in $\mathcal{A} \oplus \mathcal{M}$. We note that via the section $i : \mathcal{A} \to \mathcal{A} \oplus \mathcal{M}$ in Lemma 5.1.19, we get an additive functor

$$\mathbf{Maps}_{\mathcal{A}\oplus\mathcal{M}}(i(-), i(-)) : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathrm{Spt}_{\geqslant 0}.$$

Furthermore, by Lemma 5.1.19 again, any object in $\mathcal{A} \oplus \mathcal{M}$ is equivalent to one which is of the form $i(c)$.

Proposition 5.1.20. *There is a natural equivalence of connective spectra, functorial in both variables:*

$$\mathbf{Maps}_{\mathcal{A}\oplus\mathcal{M}}(i(c), i(c')) \cong \mathbf{Maps}_{\mathcal{A}}(c, c') \oplus \mathcal{M}(c, c').$$

Furthermore, the composition

$$i(c) \stackrel{(f_1,m_1)}{\to} i(c') \stackrel{(f_2,m_2)}{\to} i(c'')$$

in $\mathcal{A} \oplus \mathcal{M}$ can be computed as $(f_2 f_1, f_{2*} m_1 + f_1^* m_2)$.

Proof. We claim that the fiber sequence i_*(5.1.16) of functors on $\mathcal{A}^{\mathrm{op}} \times \mathcal{A}$ splits. Indeed, restricting along i amounts to setting t and t' in (5.1.15) to be the maps classifiying the zero elements and thus the desired sequence splits from by its construction. This proves the first part of the statement.

To compute the composition, we note that since the composition operation is linear with respect to the additive $\mathbb{E}_\infty$-structure, we have that

$$\begin{aligned}(f_2, m_2) \circ (f_1, m_1) &\simeq ((f_2, 0) + (0, m_2)) \circ (f_1, m_1) \\ &\simeq (f_2, 0) \circ (f_1, m_1) + (0, m_2) \circ (f_1, m_1) \\ &\simeq (f_2, 0) \circ (f_1, 0) + (f_2, 0) \circ (0, m_1) + (0, m_2) \circ (f_1, 0) + (0, m_2) \circ (0, m_1).\end{aligned}$$

Hence the claim follows from the following sequence of equivalences:

(1) $(f_2, 0) \circ (f_1, m) \simeq (f_2 \circ f_1, f_{2*} m)$,

(2) $(f_2, m) \circ (f_1, 0) \simeq (f_2 \circ f_1, f_1^* m)$, and

(3) $(0, m) \circ (0, m') \simeq 0$.

We note that (3) was shown in the course of proving Proposition 5.1.18. To prove the first claim, because of the $\mathcal{A}^{\mathrm{op}} \times \mathcal{A}$-naturality of the decomposition, we get the commutative diagram of connective spectra:

$$\begin{array}{ccc}\mathbf{Maps}_{\mathcal{A}\oplus\mathcal{M}}(i(c'), i(c'')) & \xrightarrow{\simeq} & \mathbf{Maps}_{\mathcal{A}}(c', c'') \oplus \mathcal{M}(c', c'') \\ {\scriptstyle (f_1,0)^*}\downarrow & & \downarrow {\scriptstyle f_1^* \oplus f_1^*} \\ \mathbf{Maps}_{\mathcal{A}\oplus\mathcal{M}}(i(c), i(c'')) & \xrightarrow{\simeq} & \mathbf{Maps}_{\mathcal{A}}(c, c'') \oplus \mathcal{M}(c, c'').\end{array}$$

This proves (1), while (2) follows from the commutativity of

$$\begin{array}{ccc}\mathbf{Maps}_{\mathcal{A}\oplus\mathcal{M}}(i(c), i(c')) & \xrightarrow{\simeq} & \mathbf{Maps}_{\mathcal{A}}(c, c') \oplus \mathcal{M}(c, c') \\ {\scriptstyle (f_2,0)_*}\downarrow & & \downarrow {\scriptstyle f_{2*} \oplus f_{2*}} \\ \mathbf{Maps}_{\mathcal{A}\oplus\mathcal{M}}(i(c), i(c'')) & \xrightarrow{\simeq} & \mathbf{Maps}_{\mathcal{A}}(c, c'') \oplus \mathcal{M}(c, c'').\end{array}$$

□

Remark 5.1.21. In [Dot18, Definition 1.3], Dotto defined the notion of square zero extension of Ab-enriched categories. We view our construction as an ∞-categorical analog of his construction. In contrast to his approach, our formulation does not begin by prescribing a composition law (which would be difficult to do in our setting). Rather, the composition law is a computation stemming from the presentation as a lax pullback. We can also view our definition as an ∞-categorical/spectral version of Tabuada's definition for dg categories [Tab09, Section 4].

Note that in the setting of additive categories the construction of Dotto is equivalent to ours (by Lemma 5.1.19 and Proposition 5.1.20).

Example 5.1.22. Let R be a connective $\mathbb{E}_1$-ring spectrum and M a connective R-bimodule. Out of this, we can form a connective $\mathbb{E}_1$-R-algebra spectrum $\mathrm{R}\oplus\mathrm{M}$, the square-zero extension in the sense of [Lur17a, Section 7.4.1]. The underlying R-module is given by $\mathrm{R}\oplus\mathrm{M}$. This generalizes the usual construction of square-zero extension when R and M are discrete; in this case $\mathrm{R}\oplus\mathrm{M}$ is endowed with multiplication given by:

$$(r_1, m_1), (r_2, m_2) \mapsto (r_1 r_2, r_1 m_2 + m_1 r_2).$$

Now, M determines a $\mathbf{Proj}_{\mathrm{R}}^{\mathrm{fg}}$-bimodule $\mathcal{M}$

$$\mathbf{Proj}_{\mathrm{R}}^{\mathrm{fg,op}} \times \mathbf{Proj}_{\mathrm{R}}^{\mathrm{fg}} \longrightarrow \mathrm{Spt}$$

$$(\mathrm{P}_1, \mathrm{P}_2) \mapsto \mathrm{Maps}(\mathrm{P}_1, \mathrm{R}) \otimes_{\mathrm{R}} \mathrm{M} \otimes_{\mathrm{R}} \mathrm{P}_2.$$

We claim that there is a canonical equivalence of additive ∞-categories

$$\mathbf{Proj}_{\mathrm{R}\oplus\mathrm{M}}^{\mathrm{fg}} \simeq \mathbf{Proj}_{\mathrm{R}}^{\mathrm{fg}} \oplus \mathcal{M}.$$

Indeed, by Lemma 5.1.19 $\mathbf{Perf}_{\mathrm{R}} \oplus \mathcal{M}$ is generated under finite sums and retracts by the object $(\mathrm{R}, 0)$. Now, by Theorem 5.3.1 $\mathbf{Perf}_{\mathrm{R}} \oplus \mathcal{M}$ is equivalent to the ∞-category of projective modules over $\mathbf{End}((\mathrm{R}, 0))$. Moreover, the description of the endomorphism ring is given in Proposition 5.1.20, whence we get that $\mathbf{End}((\mathrm{R}, 0)) \simeq \mathrm{R}\oplus\mathrm{M}$ as connective $\mathbb{E}_1$-rings.

5.2. **Localizing invariants.** Following the convention of [LT19] we say that a **localizing invariant** is a functor

$$\mathrm{E} : \mathbf{Cat}_{\infty}^{\mathrm{perf}} \to \mathrm{Spt},$$

which converts any exact sequence in $\mathbf{Cat}_{\infty}^{\mathrm{perf}}$ to a cofiber sequence of spectra (exact sequences in the context of stable ∞-categories are recalled in Section 2.4.2). This differs from the convention in [BGT13] where a localizing invariant was also required to commute with filtered colimits. For any scheme or ring X we set

$$\mathrm{E}(\mathrm{X}) = \mathrm{E}(\mathbf{Perf}_{\mathrm{X}}).$$

We say that E is **truncating** if for any connective $\mathbb{E}_1$-ring spectrum R, the canonical map

$$\mathrm{R} \to \pi_0(\mathrm{R})$$

induces an equivalence

$$\mathrm{E}(\mathrm{R}) \xrightarrow{\simeq} \mathrm{E}(\pi_0(\mathrm{R})).$$

Localizing invariants of additive ∞-categories are defined by applying them to the idempotent completion of the ∞-category of finite cell-modules.

Definition 5.2.1. Let E be a localizing invariant and $\mathcal{A}$ a small additive ∞-category. Then we set

$$\mathrm{E}^{\Sigma}(\mathcal{A}) := \mathrm{E}(\mathrm{Kar}\left(\mathcal{A}^{\mathrm{fin}}\right)).$$

Remark 5.2.2. Since a stable ∞-category $\mathcal{C}$ is also an additive ∞-category, we can either take $\mathrm{E}^{\Sigma}(\mathcal{C})$ or take $\mathrm{E}(\mathcal{C})$. These spectra are, in general, *not* equivalent. Furthermore, when $\mathrm{E} = \mathrm{K}$, the spectrum $\mathrm{K}^{\Sigma}(\mathcal{A})$ is also not necessarily equivalent to the direct sum K-theory $\mathrm{K}^{\oplus}(\mathcal{A})$; for example the latter is *always* a connective spectrum while $\mathrm{K}^{\Sigma}(\mathcal{A})$ need not be. This issue is further discussed in Appendix 5.2.4. Using weight structures, we will be able to identify when $\mathrm{E}(\mathcal{C}) \simeq \mathrm{E}^{\Sigma}(\mathcal{C}^{\heartsuit_w})$ as we will explain in the next section. This is the relevance of the functor E^{Σ}.

Lemma 5.2.3. *Suppose that*

$$\mathcal{B} \to \mathcal{A} \to \mathcal{C}$$

is an exact sequence of additive ∞-categories. Then for any localizing invariant E*,*

$$\mathrm{E}^{\Sigma}(\mathcal{B}) \to \mathrm{E}^{\Sigma}(\mathcal{A}) \to \mathrm{E}^{\Sigma}(\mathcal{C})$$

is a cofiber sequence of spectra.

Proof. This follows from Lemma 2.4.7 and the definition of E^{Σ}.

□

5.2.4. K*-theory of additive ∞-categories.* One of the most important localizating invariants is K-theory. Let $\mathcal{A}$ be an additive ∞-category. In this section, we clarify what it means to take $\mathrm{K}(\mathcal{A})$ and prove that in the situations we are interested in, all notions of K-theory agree. To $\mathcal{A}$ we can attach the following spectra:

(1) the **direct sum** K-theory in the style of Segal [Seg74] and revisited by Gepner-Groth-Nikolaus [GGN15, Section 8]

$$\mathrm{K}^{\oplus}(\mathcal{A}).$$

It is obtained by first taking the core of $\mathcal{A}$ to get an $\mathbb{E}_\infty$-monoid in spaces, $\mathcal{A}^{\simeq}$, whose operation is induced by direct sum. Then $\mathrm{K}^{\oplus}(\mathcal{A})$ is the *connective* spectrum obtained by taking group completion and invoking the identification between group-complete $\mathbb{E}_\infty$-monoids in spaces with connective spectra.

(2) As we have done for most of this paper we can apply the K-theory functor as in [BGT13], characterized as the universal localizing invariant, to $\mathrm{Kar}(\mathcal{A}^{\mathrm{fin}})$; we denoted this by

$$\mathrm{K}^{\Sigma}(\mathcal{A}).$$

(3) Suppose that $\mathcal{A} \subset \mathcal{C}$ is an additive subcategory of a stable ∞-category (for example, the weight-heart of a weight structure on $\mathcal{C}$). Then we can equip $\mathcal{C}$ with the structure of a **Waldhausen ∞-category** in the sense of [Bar16] by the **maximal pair structure** of [Bar16, Example 2.11] insisting that all maps are ingressives. We can then induce the structure of a Waldhausen ∞-category on $\mathcal{A}$ where the ingressives are those morphisms with cofibers in $\mathcal{A}$. To this, we can attach the **Waldhausen-Barwick** K**-theory**:

$$\mathrm{K}^{\mathrm{WB}}(\mathcal{A}),$$

as constructed in [Bar16, Part 3].

Theorem 5.2.5. *Let $\mathcal{C}$ be a boundedly weighted stable ∞-category and let $\mathcal{C}^{\heartsuit_w} \subset \mathcal{C}$ be its weight heart. Then:*

(1) There are canonical equivalences of connective spectra

$$\mathrm{K}^{\oplus}(\mathcal{C}^{\heartsuit_w}) \simeq \mathrm{K}^{\mathrm{WB}}(\mathcal{C}^{\heartsuit_w}) \simeq \mathrm{K}^{\mathrm{WB}}(\mathcal{C})$$

(2) There is a canonical morphism

$$\mathrm{K}^{\mathrm{WB}}(\mathcal{C}^{\heartsuit_w}) \to \mathrm{K}^{\Sigma}((\mathcal{C}^{\heartsuit_w})^{\mathrm{fin}})$$

which identifies as a connective cover.

Proof. Using [Fon18, Theorem 5.1] or [Hel20, Theorem A.15], we have an equivalence of connective spectra
$$\mathrm{K}^{\mathrm{WB}}(\mathcal{C}) \simeq \mathrm{K}^{\mathrm{WB}}(\mathcal{C}^{\heartsuit_w}).$$
Using [Hel20, Proposition A.19], noting that ingressives are split in $\mathcal{C}^{\heartsuit_w}$, we have a further identification:
$$\mathrm{K}^{\oplus}(\mathcal{C}^{\heartsuit_w}) \simeq \mathrm{K}^{\mathrm{WB}}(\mathcal{C}^{\heartsuit_w}).$$

For the second statement we note that, by Theorem 3.2.3, we have an equivalence of nonconnective spectra:
$$\mathrm{K}(\mathcal{C}) \simeq \mathrm{K}((\mathcal{C}^{\heartsuit_w})^{\mathrm{fin}}).$$
On the other hand, the connective version of K-theory of the stable ∞-category $\mathcal{C}$ in the sense of [BGT13] is, by construction, the same as $\mathrm{K}^{\mathrm{WB}}(\mathcal{C})$ with the maximal pair structure. Therefore, we have a natural map to the nonconnective K-theory of [BGT13]
$$\mathrm{K}^{\mathrm{WB}}(\mathcal{C}) \to \mathrm{K}(\mathcal{C})$$
which witnesses a connective cover. Therefore, the map induced by the equivalence of the first part:
$$\mathrm{K}^{\mathrm{WB}}(\mathcal{C}^{\heartsuit_w}) \simeq \mathrm{K}^{\mathrm{WB}}(\mathcal{C}) \to \mathrm{K}((\mathcal{C}^{\heartsuit_w})^{\mathrm{fin}}),$$
is indeed a connective cover. □

5.3. **Morita theory for additive ∞-categories.** In the course of proving the main theorem of this section we will reduce questions about abstract additive ∞-categories to those of the form $\mathbf{Proj}^{\mathrm{fg}}_{\mathrm{R}}$. The next result is the main tool that allows us to do so. This is an analog of the well-known result of Schwede and Shipley [SS03, Theorem 3.1.1] (see also [Lur17a, Section 7.1.2]) in the context of additive ∞-categories. The additive ∞-category of finite projective modules over the sphere spectrum $\mathbf{Proj}^{\mathrm{fg}}_{\mathbb{S}}$ is an idempotent complete additive ∞-category and thus is an object of $\mathbf{Cat}^{\mathrm{Kar}}_{\infty}$. We denote the slice category under $\mathbf{Proj}^{\mathrm{fg}}_{\mathbb{S}}$ by:
$$\mathbf{Cat}^{\mathrm{Kar}}_{\infty,\mathbf{Proj}^{\mathrm{fg}}_{\mathbb{S}}/}.$$
Its objects are additive functors $\mathbf{Proj}^{\mathrm{fg}}_{\mathbb{S}} \to \mathcal{A}$; equivalently this picks an object $\mathrm{R} = \mathrm{F}(\mathbb{S})$ of $\mathcal{A}$. Hence we can regard $\mathbf{Cat}^{\mathrm{Kar}}_{\infty,\mathbf{Proj}^{\mathrm{fg}}_{\mathbb{S}}/}$ as the ∞-category of idempotent complete additive ∞-categories with a chosen object.

Theorem 5.3.1. *There is a functor*
$$\Theta : \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Spt}^{\mathrm{cn}}) \to \mathbf{Cat}^{\mathrm{Kar}}_{\infty,\mathbf{Proj}^{\mathrm{fg}}_{\mathbb{S}}/}$$
that sends a ring R *to the object in the undercategory:*
$$\mathbf{Proj}^{\mathrm{fg}}_{\mathbb{S}} \to \mathbf{Proj}^{\mathrm{fg}}_{\mathrm{R}}.$$
Then,

(1) the functor Θ is fully faithful;

(2) an object $\mathbf{Proj}^{\mathrm{fg}}_{\mathbb{S}} \xrightarrow{\mathrm{F}} \mathcal{A}$ in $\mathbf{Cat}^{\mathrm{Kar}}_{\infty,\mathbf{Proj}^{\mathrm{fg}}_{\mathbb{S}}/}$ is in the essential image of Θ if and only if $\mathcal{A}$ is generated by $\mathrm{F}(\mathbb{S})$ under finite sums and retracts. In this case, F *is equivalent to $\Theta(\mathbf{End}_{\mathcal{A}}(\mathrm{F}(\mathbb{S})))$. In other words,* F *is equivalent to*
$$\mathbf{Proj}^{\mathrm{fg}}_{\mathbb{S}} \to \mathbf{Proj}^{\mathrm{fg}}_{\mathbf{End}_{\mathcal{A}}(\mathrm{F}(\mathbb{S}))},$$
given by extension along the map of conenective $\mathbb{E}_1$-ring spectra: $\mathbb{S} \to \mathbf{End}_{\mathcal{A}}(\mathrm{F}(\mathbb{S}))$.

Proof. To construct the functor consider the cocartesian fibration defining the functor Consider a cocartesian fibration

$$p : \mathbf{RMod} \to \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Spt}^{\mathrm{cn}})$$

classified by the functor sending a ring R to the ∞-category $\mathbf{RMod}_{\mathrm{R}}$ (see [Lur17a, Construction 4.8.3.24]). Consider the full subcategory $\mathbf{Proj}^{\mathrm{fg}} \subset \mathbf{RMod}$ consisting of those M over R that are finitely generated projective. Since projective modules are closed under pullback, the composite

$$\mathbf{Proj}^{\mathrm{fg}} \to \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Spt}^{\mathrm{cn}})$$

is also a cocartesian fibration classified by a functor

$$\mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Spt}^{\mathrm{cn}}) \to \mathbf{Cat}_{\infty}^{\mathrm{Kar}} \to \mathbf{Cat}_{\infty}.$$

This yields a desired functor

$$\Theta : \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Spt}^{\mathrm{cn}}) \to \mathbf{Cat}^{\mathrm{Kar}}_{\infty,\mathbf{Proj}^{\mathrm{fg}}_{\mathbb{S}}/}.$$

To prove (1) note that the canonical functor $\widehat{\mathbf{Proj}^{\mathrm{fg}}_{\mathrm{R}}} \to \mathbf{RMod}_{\mathrm{R}}$ is an equivalence (combine Corollary 3.2.9 and [Lur17b, Proposition 5.3.6.2]), so the composite functor

$$\mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Spt}^{\mathrm{cn}}) \xrightarrow{\Theta} \mathbf{Cat}^{\mathrm{Kar}}_{\infty,\mathbf{Proj}^{\mathrm{fg}}_{\mathbb{S}}/} \xrightarrow{\widehat{(-)}} \mathcal{P}\mathrm{r}^{\mathrm{St}}_{\mathrm{Spt}/},$$

is equivalent to the functor sending R to $\mathbf{RMod}_{\mathrm{R}}$. Combining [Lur17a, Theorem 4.8.5.5] and [Lur17a, Proposition 4.8.2.18] we see that this functor is fully faithful. Hence to prove that Θ is fully faithful it suffices to prove that the induced map on mapping spaces

$$\mathrm{Maps}_{\mathbf{Cat}^{\mathrm{Kar}}_{\infty,\mathbf{Proj}^{\mathrm{fg}}_{\mathbb{S}}/}}(\mathbf{Proj}^{\mathrm{fg}}_{\mathrm{R}}, \mathbf{Proj}^{\mathrm{fg}}_{\mathrm{S}}) \to \mathrm{Maps}_{\mathcal{P}\mathrm{r}^{\mathrm{St}}_{\mathrm{Spt}/}}(\mathbf{RMod}_{\mathrm{R}}, \mathbf{RMod}_{\mathrm{S}})$$

is an equivalence. This map is an epimorphism since the functor $\widehat{(\Theta(-))}$ was already known to be fully faithful; it suffices to prove that this map is a subspace inclusion. The above map fits as the upper horizontal map in the following diagram where the vertical arrows define fiber sequences (from how mapping spaces of undercategories are computed):

$$\begin{array}{ccc}
\mathrm{Maps}_{\mathbf{Cat}^{\mathrm{Kar}}_{\infty,\mathbf{Proj}^{\mathrm{fg}}_{\mathbb{S}}/}}(\mathbf{Proj}^{\mathrm{fg}}_{\mathrm{R}}, \mathbf{Proj}^{\mathrm{fg}}_{\mathrm{S}}) & \longrightarrow & \mathrm{Maps}_{\mathcal{P}\mathrm{r}^{\mathrm{St}}_{\mathrm{Spt}/}}(\mathbf{RMod}_{\mathrm{R}}, \mathbf{RMod}_{\mathrm{S}}) \\
\downarrow & & \downarrow \\
\mathrm{Maps}_{\mathbf{Cat}^{\mathrm{Kar}}_{\infty}}(\mathbf{Proj}^{\mathrm{fg}}_{\mathrm{R}}, \mathbf{Proj}^{\mathrm{fg}}_{\mathrm{S}}) & \longrightarrow & \mathrm{Maps}_{\mathcal{P}\mathrm{r}^{\mathrm{St}}}(\mathbf{RMod}_{\mathrm{R}}, \mathbf{RMod}_{\mathrm{S}}) \\
\downarrow & & \downarrow \\
\mathrm{Maps}_{\mathbf{Cat}^{\mathrm{Kar}}_{\infty}}(\mathbf{Proj}^{\mathrm{fg}}_{\mathbb{S}}, \mathbf{Proj}^{\mathrm{fg}}_{\mathrm{S}}) & \longrightarrow & \mathrm{Maps}_{\mathcal{P}\mathrm{r}^{\mathrm{St}}}(\mathrm{Spt}, \mathbf{RMod}_{\mathrm{S}})
\end{array}$$

Now, combining the universal properties [Lur17a, Proposition 5.3.6.2] and [Lur18, Proposition C.1.1.7] we obtain that the middle and the bottom horizontal functors are subspace inclusions. Therefore, we conclude that the upper horizontal functor is also a subspace inclusion.

Now we prove (2). If $\mathcal{A} = \mathbf{Proj}^{\mathrm{fg}}_{\mathrm{R}}$ for a connective $\mathbb{E}_1$-ring R, then $\mathrm{R} = \mathrm{F}(\mathbb{S})$ generates $\mathbf{Proj}^{\mathrm{fg}}_{\mathrm{R}}$ under finite sums and retracts by definition of this category (see Example 2.4.6). It suffices to prove the converse implication. Assume that $\mathcal{A}$ is generated, as an additive ∞-category, by $\mathrm{F}(\mathbb{S})$. Let $\mathrm{R} := \mathbf{End}(\mathrm{F}(\mathbb{S}))$ be the endomorphism $\mathbb{E}_1$-ring spectrum. We have a functor

$$\mathbf{Maps}(\mathrm{F}(\mathbb{S}), -) : \mathcal{A} \to \mathbf{RMod}_{\mathrm{R}}$$

which is fully faithful since it is fully faithful on $\mathrm{F}(\mathbb{S})$ and $\mathrm{F}(\mathbb{S})$ generates $\mathcal{A}$ under direct sums and retracts. To conclude, we note that functor sends $\mathrm{F}(\mathbb{S})$ to R and its essential image consists precisely of finite direct sums of R and retracts, i.e., the subcategory $\mathbf{Proj}^{\mathrm{fg}}_{\mathrm{R}}$.

$\square$

5.3.2. *Proof of DGM theorem for additive ∞-categories.* We now come to our main result in the setting of additive ∞-categories.

Theorem 5.3.3. *Let $f : \mathcal{A} \to \mathcal{B}$ be a nilpotent extension of small additive ∞-categories and let E be a truncating invariant. Then, we have an induced equivalence of spectra*

$$\mathrm{E}^{\Sigma}(\mathcal{A}) \to \mathrm{E}^{\Sigma}(\mathcal{B}).$$

Proof. We will explain the following commutative diagram of *small* additive ∞-categories

$$\begin{array}{ccccc} \mathcal{A} & \longrightarrow & \mathcal{A}^{\mathrm{big}} & \longrightarrow & (\mathcal{A}^{\mathrm{big}}/\mathcal{A})^{\mathrm{Kar}} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{B} & \longrightarrow & \mathcal{B}^{\mathrm{big}} & \longrightarrow & (\mathcal{B}^{\mathrm{big}}/\mathcal{B})^{\mathrm{Kar}}. \end{array} \tag{5.3.4}$$

For a small additive ∞-category $\mathcal{E}$, we consider

$$\mathcal{E}^{\mathrm{big}} \subset \mathrm{Ind}(\mathcal{E}),$$

which is the additive subcategory closed under retracts generated by

$$\mathrm{G}_{\mathcal{E}} = \bigoplus_{x \in \mathrm{Obj}(\mathcal{E})} x.$$

This is evidently a small additive ∞-category.

The canonical functor $\mathcal{E} \to \mathrm{Ind}(\mathcal{E})$ factors through $\mathcal{E}^{\mathrm{big}}$ since the latter category is closed under retracts, whence contains the Yoneda image. Note that this construction is functorial in small additive ∞-categories and additive functors. This explains the left square. The right square is just the induced map on cofibers, i.e., on Verdier quotients in the sense explained in Section 2.2.

We note the following facts:

(1) By Theorem 5.3.1(2) we have an equivalence of additive ∞-categories

$$\mathcal{A}^{\mathrm{big}} \simeq \mathbf{Proj}^{\mathrm{fg}}_{\mathbf{End}_{\mathcal{A}^{\mathrm{big}}}(\mathrm{G}_{\mathcal{A}})},$$

(2) the additive ∞-category $\mathcal{A}^{\mathrm{big}}/\mathcal{A}$ is generated by the image of $\mathrm{G}_{\mathcal{A}}$ under finite sums and retracts.

Therefore applying Theorem 5.3.1(2) again:

$$(\mathcal{A}^{\mathrm{big}}/\mathcal{A})^{\mathrm{Kar}} \simeq \mathbf{Proj}^{\mathrm{fg}}_{\mathbf{End}_{\mathcal{A}^{\mathrm{big}}/\mathcal{A}}(\mathrm{G}_{\mathcal{A}})},$$

Since f is a nilpotent extension of additive ∞-categories (and is thus essentially surjective), we have that $\mathrm{G}_{\mathcal{B}}$ is a retract of $f(\mathrm{G}_{\mathcal{A}})$ and in particular $f(\mathrm{G}_{\mathcal{A}})$ generates $\mathcal{B}^{\mathrm{big}}$ under finite sums and retracts. Therefore, the analogous equivalences also hold for $f(\mathrm{G}_{\mathcal{A}})$ in $\mathcal{B}^{\mathrm{big}}$ and in $(\mathcal{B}^{\mathrm{big}}/\mathcal{B})^{\mathrm{Kar}}$. Under these equivalences the middle and the right map of (5.3.4) correspond to base change functors along maps of $\mathbb{E}_1$-rings.

Now, the map

$$\pi_0\mathbf{End}_{\mathcal{A}^{\mathrm{big}}}(\mathrm{G}_{\mathcal{A}}) \to \pi_0\mathbf{End}_{\mathcal{B}^{\mathrm{big}}}(f(\mathrm{G}_{\mathcal{A}}))$$

is a nilpotent extension of rings. Indeed, $\pi_0\mathbf{End}_{\mathcal{A}^{\mathrm{big}}}(\mathrm{G}_{\mathcal{A}})$ can be expressed as an infinite ring of matrices which are *column finite* with entries in $\pi_0 \operatorname{Maps}_{\mathcal{A}}(x_i, x_j), x_i, x_j \in \mathcal{A}$, and similarly for $\pi_0\mathbf{End}_{\mathcal{B}^{\mathrm{big}}}(f(\mathrm{G}_{\mathcal{A}}))$. The kernel of this ring map consists of (infinite) matrices which are column finite and whose entries are elements $s \in \pi_0 \operatorname{Maps}_{\mathcal{A}}(x_i, x_j)$ such that $f(s) = 0$. From the definition of a nilpotent extension of additive ∞-categories, this a nilpotent ideal. Moreover, the same is true for the kernel of the map

$$\pi_0\mathbf{End}_{\mathcal{A}^{\text{big}}/\mathcal{A}}(\mathrm{G}_{\mathcal{A}}) \to \pi_0\mathbf{End}_{\mathcal{B}^{\text{big}}/\mathcal{B}}(f(\mathrm{G}_{\mathcal{A}}))$$

by Corollary 2.2.4(2) which expresses each of the rings as a quotient of column-finite matrices modulo the ideal of matrices with only finitely many nonzero entries.

Now E^{Σ} applied to the middle or the right vertical functor of (5.3.4) is an equivalence by the assumption that E is truncating. Since it is a localizing invariant it also induces an equivalence after applying it to $\mathcal{A} \to \mathcal{B}$ which yields the claim. □

Here is a couple of corollaries. The next one is an ∞-categorical version of Dotto's theorem [Dot18]. This generalizes the split-exact case of his theorem by Remark 5.1.21.

Corollary 5.3.5. *Suppose that $\mathcal{A}$ is a small additive ∞-category and $\mathcal{M}$ is an $\mathcal{A}$-bimodule. Then for any truncating invariant* E *we have that*

$$\mathrm{E}^{\Sigma}(\mathcal{A}) \simeq \mathrm{E}^{\Sigma}(\mathcal{A} \oplus \mathcal{M}).$$

In particular if we set $\mathrm{E} = \mathrm{K}^{\mathrm{inv},\Sigma}$*, then we get a cartesian square:*

$$\begin{array}{ccc} \mathrm{K}^{\Sigma}(\mathcal{A} \oplus \mathcal{M}) & \longrightarrow & \mathrm{TC}^{\Sigma}(\mathcal{A} \oplus \mathcal{M}) \\ \big\downarrow & & \big\downarrow \\ \mathrm{K}^{\Sigma}(\mathcal{A}) & \longrightarrow & \mathrm{TC}^{\Sigma}(\mathcal{A}). \end{array}$$

The next corollary will be important for the sequel.

Corollary 5.3.6. *Let $\mathcal{A}$ be an addititive ∞-category. Then the functor $\mathcal{A} \to h\mathcal{A}$ induces an equivalence, for any truncating invariant* E*,*

$$\mathrm{E}^{\Sigma}(\mathcal{A}) \simeq \mathrm{E}^{\Sigma}(h\mathcal{A}).$$

In particular if we set $\mathrm{E} = \mathrm{K}^{\mathrm{inv},\Sigma}$*, then we get a cartesian square:*

$$\begin{array}{ccc} \mathrm{K}^{\Sigma}(\mathcal{A}) & \longrightarrow & \mathrm{TC}^{\Sigma}(\mathcal{A}) \\ \big\downarrow & & \big\downarrow \\ \mathrm{K}^{\Sigma}(h(\mathcal{A})) & \longrightarrow & \mathrm{TC}^{\Sigma}(h(\mathcal{A})). \end{array}$$

Remark 5.3.7. TC is not a finitary invariant. Indeed, as in [LMT20, Section 3.1], [BCM20, Section 2.4], we can write the subcategory of p^{∞}-torsion objects in $\mathbf{Perf}_{\mathbb{Z}}$ as a colimit of categories of modules:

$$\operatorname{colim} \mathbf{Mod}_{\mathbb{Z}/p^n\mathbb{Z}}(\mathbf{Perf}_{\mathbb{Z}}).$$

However, TC does not preserve this colimit as explained in [LMT20, Remark 3.27]. Thus we cannot prove DGM for a nilpotent extension of additive ∞-categories $\mathcal{A} \to \mathcal{B}$ simply by applying the original DGM theorem to the maps

$$\mathbf{End}_{\mathcal{A}}(x) \to \mathbf{End}_{\mathcal{B}}(x),$$

as x varies through the objects of $\mathcal{A}$ (and thus $\mathcal{B}$) and then passing to the colimit. We note, however, the following amusing corollary of Theorem 5.3.3.

Corollary 5.3.8. *The functor*

$$\mathbf{Cat}_{\infty}^{\text{add}} \to \mathrm{Spt} \qquad \mathcal{A} \mapsto \mathrm{Fib}(\mathrm{TC}^{\Sigma}(\mathcal{A}) \to \mathrm{TC}^{\Sigma}(h\mathcal{A}))$$

commutes with filtered colimits.

Proof. After Theorem 5.3.3, this follows from the fact that the K-theory functor, the functor $\mathcal{A} \mapsto h(\mathcal{A})$ and the functor $\mathcal{A} \mapsto \mathrm{Kar}(\mathcal{A}^{\mathrm{fin}})$ preserve filtered colimits.

□

5.3.9. *Negative K-theory of additive categories.* Finally we explain the following theorem whose proof is very similar to that of Theorem 5.3.3.

Theorem 5.3.10. *Let* E *be a localizing invariant such that* $\pi_0\mathrm{E}$ *is invariant under nilpotent extensions, i.e.*

$$\pi_0\mathrm{E}(\mathcal{A}) \to \pi_0\mathrm{E}(\mathcal{B})$$

is an isomorphism for any nilpotent extension of small additive ∞*-categories* $\mathcal{A} \to \mathcal{B}$. *Then the same holds for* $\pi_i\mathrm{E}(-)$ *for any* $i < 0$.

Proof. For an additive ∞-category $\mathcal{E}$ we denote by $\mathcal{E}^+$ the minimal full subcategory of $\widehat{\mathcal{E}}$ containing $\mathcal{E}$ closed under countable direct sums. This is a small category. Note that $\mathrm{E}^\Sigma(\mathcal{E}^+)$ is contractible. Indeed, the functor $\mathrm{F} : \mathcal{E} \to \mathcal{E}$ defined by the formula

$$\mathrm{X} \mapsto \bigoplus_{i \geqslant 0} \mathrm{X}$$

is additive and satisfies

$$\mathrm{id}_{\mathcal{E}} \oplus \mathrm{F} \simeq \mathrm{F}.$$

Thus we also have

$$\mathrm{id}_{\mathcal{E}^{\mathrm{fin}}} \oplus \mathrm{F}^{\mathrm{fin}} \simeq \mathrm{F}^{\mathrm{fin}}$$

and by the additivity theorem

$$\mathrm{E}^\Sigma(\mathrm{id}_{\mathcal{E}^{\mathrm{fin}}}) \simeq \mathrm{E}^\Sigma(\mathrm{F}^{\mathrm{fin}}) - \mathrm{E}^\Sigma(\mathrm{F}^{\mathrm{fin}}) \simeq 0.$$

Using the fact that $\Sigma^n\mathrm{E}$ is a localizing invariant for any $n \in \mathbb{Z}$ and induction on i we reduce to proving the statement for $i = -1$. Let $\mathcal{A} \to \mathcal{B}$ be a nilpotent extension. Using the construction above we obtain a commutative diagram

$$\begin{array}{ccccc} \mathcal{A} & \longrightarrow & \mathcal{A}^+ & \longrightarrow & (\mathcal{A}^\omega/\mathcal{A})^{\mathrm{Kar}} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{B} & \longrightarrow & \mathcal{B}^+ & \longrightarrow & (\mathcal{B}^+/\mathcal{B})^{\mathrm{Kar}}. \end{array}$$

After applying $\mathrm{E}^\Sigma(-)$ this induces a diagram in spectra whose rows are fiber sequences. And by the observation above $\mathrm{E}^\Sigma(\mathcal{A}^+)$ and $\mathrm{E}^\Sigma(\mathcal{B}^+)$ are contractible. Hence we have a commutative diagram

$$\begin{array}{ccc} \pi_0((\mathcal{A}^+/\mathcal{A})^{\mathrm{Kar}}) & \xrightarrow{\simeq} & \pi_{-1}\mathrm{E}(\mathcal{A}) \\ \downarrow & & \downarrow \\ \pi_0((\mathcal{B}^+/\mathcal{B})^{\mathrm{Kar}}) & \xrightarrow{\simeq} & \pi_{-1}\mathrm{E}(\mathcal{B}) \end{array} \tag{5.3.11}$$

Exactly the same computation as in the end of the proof of Theorem 5.3.3 shows that the map

$$(\mathcal{A}^+/\mathcal{A})^{\mathrm{Kar}} \to (\mathcal{B}^+/\mathcal{B})^{\mathrm{Kar}}$$

is a nilpotent extension. Hence both vertical maps in the diagram (5.3.11) are isomorphisms. □

By [Bon10, Theorem 5.3.1] the map $\pi_0\mathrm{K}^\Sigma(\mathcal{A}) \to \pi_0\mathrm{K}^\Sigma(h(\mathcal{A}))$ is an isomorphism for any additive ∞-category $\mathcal{A}$. Moreover since K-theory is finitary and $\pi_0\mathrm{K}^\Sigma$ is nilinvarint on rings (see [Wei13, Lemma II.2.2]), so K-theory satisfies the assumptions of the theorem. This way we obtain the following generalization of [BGT13, Theorem 9.53].

Corollary 5.3.12. *The map* $\pi_i \mathrm{K}^\Sigma(\mathcal{A}) \to \pi_i \mathrm{K}^\Sigma(\mathcal{B})$ *is an isomorphism for any nilpotent extension* $\mathcal{A} \to \mathcal{B}$ *and* $i \leqslant 0$.

Remark 5.3.13. The main result of [AGH19] says that the K-theory of a stable ∞-category with a noetherian t-structure is a connective spectrum. Now combining that result with Corollary 5.3.12 in the case $\mathcal{C} = \mathbf{DM}_{\mathrm{gm}}(k; \mathrm{R})$ yields that existence of a noetherian motivic t-structure on $\mathbf{DM}_{\mathrm{gm}}(k; \mathrm{R})$ implies vanishing of the additive K-theory groups

$$\mathrm{K}^\Sigma_{-n}(\mathbf{Chow}(k; \mathrm{R}))$$

for $n > 0$.

5.4. **The DGM theorem for weighted stable ∞-categories and examples.** So far we have only discussed additive ∞-categories and proved a DGM theorem in that context. However, the examples we are mostly interested in come from the context of stable ∞-categories, and their K-theory. Note that every stable ∞-category is also additive. The K-theory of a stable ∞-category is fundamentally different from its K-theory taken as an additive ∞-category in the sense of our previous section, i.e. we should not expect the two spectra $\mathrm{K}^\Sigma(\mathcal{C})$ and $\mathrm{K}(\mathcal{C})$ to be equivalent. Hence, our results do not directly apply.

Moreover, we do not quite know how to define nilpotent extensions of general stable ∞-categories. Note that if we regard an exact functor of stable ∞-categories as an additive functor of additive ∞-categories, then the following is a non-example of nilpotent extensions (in the sense of Definition 5.1.1).

Example 5.4.1. The morphism of connective $\mathbb{E}_\infty$-rings $\mathbb{S} \to \mathbb{Z}$ defines, via base change, a functor $\mathbf{Perf}_{\mathbb{S}} \to \mathbf{Perf}_{\mathbb{Z}}$. This is not a nilpotent extension of additive ∞-categories: the periodicity theorem in chromatic homotopy theory tells us that for any non-trivial finite spectrum X with torsion homology there are positive degree maps $\mathrm{X} \xrightarrow{f} \Sigma^d \mathrm{X}$ that are trivial in homology such that f^n is non-trivial for any $n > 0$.

On the other hand, we could try to relax the nilpotence condition by focusing on generators. This seems to lead to a wrong notion in the sense that one should not expect a DGM theorem for it, as explained in the next example.

Example 5.4.2. Let k be any ring and consider the stable ∞-category $\mathbf{Perf}_{k^{\times 2}}$, which is generated by the two projective modules $\mathrm{P}_1, \mathrm{P}_2$ given by the images of the two non-trivial idempotents in the ring $k^{\times 2}$. The endomorphism ring spectrum of either of them is k. Identifying $k^{\times 2}$ with diagonal matrices we get a ring homomorphism $k^{\times 2} \to \mathrm{M}_2(k)$ which induces a functor $\mathbf{Perf}_{k^{\times 2}} \to \mathbf{Perf}_{\mathrm{M}_2(k)}$. This functor is essentially surjective and induces an equivalence on endomorphism rings of each of the generators. However, the DGM theorem does not hold for the map $k^{\times 2} \to \mathrm{M}_2(k)$.

What we do instead is go through weight structures to define nilpotent extensions and show that the results of the previous section prove new cases of the DGM theorem in the context of stable ∞-categories.

Definition 5.4.3. Let $(\mathcal{A}, w), (\mathcal{B}, w')$ be weighted ∞-categories. A weight exact functor $f : (\mathcal{A}, w) \to (\mathcal{B}, w')$ is said to be a **nilpotent extension** if the functor of additive ∞-categories

$$f : \mathcal{A}^{\heartsuit_w} \to \mathcal{B}^{\heartsuit_{w'}}$$

is a nilpotent extension in the sense of Definition 5.1.1.

Theorem 5.4.4. *Let* $f : (\mathcal{A}, w) \to (\mathcal{B}, w')$ *be a nilpotent extension of boundedly weighted stable* ∞*-categories and let* E *be a truncating invariant. Then we have an induced equivalence of spectra*

$$\mathrm{E}(\mathcal{A}) \to \mathrm{E}(\mathcal{B}).$$

Proof. This follows immediately from Theorem 5.3.3, and Theorem 3.2.3 which gives identifications

$$\mathrm{E}(\mathcal{A}) \simeq \mathrm{E}^{\Sigma}(\mathcal{A}^{\heartsuit_w}) \qquad \mathrm{E}(\mathcal{B}) \simeq \mathrm{E}^{\Sigma}(\mathcal{B}^{\heartsuit_{w'}}).$$

□

Remark 5.4.5. We note that although Theorem 5.4.4 is proved for "absolute" localizing invariants, its proof carries over to truncating invariants of k-linear boundedly weighted stable ∞-categories where k is a connective $\mathbb{E}_\infty$-ring spectrum; we refer to [LT19, Remarks 1.18, 3.4] for more discussions on this point.

Corollary 5.4.6. *Let $f : (\mathcal{A}, w) \to (\mathcal{B}, w')$ be a nilpotent extension of boundedly weighted stable ∞-categories. Then we have an induced cartesian square of spectra:*

$$\begin{array}{ccc} \mathrm{K}(\mathcal{A}) & \longrightarrow & \mathrm{TC}(\mathcal{A}) \\ \downarrow & & \downarrow \\ \mathrm{K}(\mathcal{B}) & \longrightarrow & \mathrm{TC}(\mathcal{B}). \end{array}$$

5.4.7. *Specialization of results.* We now discuss the examples that appear in the introduction.

Example 5.4.8. Recall that for any boundedly weighted stable ∞-category $\mathcal{C}$ one has a functor

$$\mathcal{C} \simeq (\mathcal{C}^{\heartsuit_w})^{\mathrm{fin}} \to (h\mathcal{C}^{\heartsuit_w})^{\mathrm{fin}}$$

which we call the **weight complex functor** (see 3.2.10). This induces an equivalence on the homotopy categories of the heart, so it is a nilpotent extension of weighted categories. The ∞-category $(h\mathcal{C}^{\heartsuit_w})^{\mathrm{fin}}$ can also be described as the *homotopy category of bounded complexes* in $h\mathcal{C}^{\heartsuit_w}$, i.e. the localization of the 1-category of complexes by the set of homotopy equivalences. Therefore, Theorem 5.3.3 tells us that for any truncating invariant E, we have an induced equivalence

$$\mathrm{E}(\mathcal{C}) \simeq \mathrm{E}((h\mathcal{C}^{\heartsuit_w})^{\mathrm{fin}}) = \mathrm{E}^{\Sigma}(h\mathcal{C}^{\heartsuit_w}).$$

Example 5.4.9. Now we specialize Corollary 5.4.6 to the setting of Theorem 3.3.4. Recall that $h\mathbf{Chow}_\infty$ is equivalent to the classical additive category of Chow motives **Chow**, so the functor

$$\mathbf{DM}_{\mathrm{gm}}(k; \mathrm{R}) \to (\mathbf{Chow}(k; \mathrm{R}))^{\mathrm{fin}}$$

is a nilpotent extension. Thus we obtain the Corollary 1.4.6 from the introduction.

Example 5.4.10. Now, let k be a discrete ring and G be an embeddable linearly reductive group scheme over k. Suppose that R, S are connective $\mathbb{E}_\infty$-k-algebras endowed with G-actions and $f : \mathrm{R} \to \mathrm{S}$ is a G-equivariant map over k. Assume further that $\pi_0(\mathrm{R}) \overset{\pi_0(f)}{\to} \pi_0(\mathrm{S})$ is a surjection with nilpotent kernel. Now Theorem 3.3.13(1) endows both $\mathbf{Perf}_{[\operatorname{Spec} \mathrm{R}/\mathrm{G}]}$ and $\mathbf{Perf}_{[\operatorname{Spec} \mathrm{S}/\mathrm{G}]}$ with weight structures and the induced functor $f^* : \mathbf{Perf}_{[\operatorname{Spec} \mathrm{R}/\mathrm{G}]} \to \mathbf{Perf}_{[\operatorname{Spec} \mathrm{S}/\mathrm{G}]}$ is weight exact by Theorem 3.3.13(2). Below we prove that f^* is a nilpotent extension of boundedly weighted stable ∞-categories. Applying Theorem 5.4.4 to this functor we obtain Corollary 1.4.7 from the introduction.

Proposition 5.4.11. *With the notation of Example 5.4.10, the functor*

$$f^* : \mathbf{Perf}_{[\operatorname{Spec} \mathrm{R}/\mathrm{G}]} \to \mathbf{Perf}_{[\operatorname{Spec} \mathrm{S}/\mathrm{G}]}$$

is a nilpotent extension of boundedly weighted stable ∞-categories.

Proof. Denote by

$$p_{\mathrm{R}} \colon [\operatorname{Spec} \mathrm{R}/\mathrm{G}] \to \mathrm{BG},$$
$$p_{\mathrm{S}} \colon [\operatorname{Spec} \mathrm{S}/\mathrm{G}] \to \mathrm{BG}$$

the canonical projections and by $\overline{f}$ the map $[\operatorname{Spec} S/G] \to [\operatorname{Spec} R/G]$ induced by f. By construction, the hearts of the weight structures on $\mathbf{Perf}_{[\operatorname{Spec} R/G]}$ and $\mathbf{Perf}_{[\operatorname{Spec} S/G]}$ are generated under finite sums and retracts by pullbacks of finite G-equivariant k-modules, along p_R and p_S, respectively.

By descent we may identify $\operatorname{Maps}_{[\operatorname{Spec} R/G]}(p_R^* V, p_R^* W)$ with $\operatorname{Maps}_R(V, W)^G$ (and similarly for S). Note that the map

$$\pi_0 \operatorname{Maps}_R(V, W)^G \to \pi_0 \operatorname{Maps}_S(f^* V, f^* W)^G$$

is surjective since G is linearly reductive.

Let n be an integer such that $I^n = 0$ where $I = \operatorname{Ker}(\pi_0(R) \to \pi_0(S))$. Up to adding direct summands the elements in the kernel of

$$\pi_0 \operatorname{Maps}_R(V, W) \to \pi_0 \operatorname{Maps}_S(f^* V, f^* W)$$

are defined by matrices with coefficients in I. Therefore, for any sequence of composable morphisms $f_1, \cdots, f_n$ in $\mathbf{Proj}_R^{fg}$ such that $f^*(f_i)$ is trivial in $\mathbf{Proj}_S^{fg}$, we get that $f_1 \circ \cdots \circ f_n = 0$. Now since $\pi_0 \operatorname{Maps}_R(V, W)^G$ is a subgroup in $\pi_0 \operatorname{Maps}_R(V, W)$, the same is true for composable sequences of morphisms in $\mathbf{Perf}_{[\operatorname{Spec} R/G]}^{\heartsuit_w}$ whose base change to S is trivial. So, the second and the third parts of Definition 5.1.1 are satisfied.

Since $p_R \circ \overline{f} = p_S$, the restriction of $\overline{f}^*$ to the hearts is essentially surjective up to retracts. Any idempotent $p \in \pi_0 \operatorname{End}_{[\operatorname{Spec} S/G]}(f^* V)$ can be lifted along a nilpotent extension of rings

$$\pi_0 \operatorname{End}_{[\operatorname{Spec} R/G]}(V) \to \pi_0 \operatorname{End}_{[\operatorname{Spec} S/G]}(f^* V),$$

so $\overline{f}^*$ is essentially surjective and the first part of Definition 5.1.1 is also satisfied. □

Applying Theorem 5.4.4 to the nilpotent extension from Proposition 5.4.11 we obtain a proof of Corollary 1.4.7. Recall that given a localizing invariant E and a connective $\mathbb{E}_\infty$-k-algebra R with a G-action, we set

$$E^G(R) := E(\mathbf{Perf}_{[\operatorname{Spec} R/G]}).$$

Corollary 5.4.12. *Let* $R \to S$ *be a* G*-equivariant map of connective* $\mathbb{E}_\infty$*-*k*-algebras. Then for any truncating invariant* E *the natural map*

$$E^G(R) \to E^G(S)$$

is an equivalence.

In particular, the square

$$\begin{array}{ccc} K^G(R) & \longrightarrow & TC^G(R) \\ \downarrow & & \downarrow \\ K^G(S) & \longrightarrow & TC^G(S) \end{array}$$

is cartesian.

Example 5.4.13. In the notation of Example 5.4.10, let R be a connective $\mathbb{E}_\infty$-k-algebra with a G-action. Then the map $[\operatorname{Spec} \pi_0(R)/G] \to [\operatorname{Spec} R/G]$ satisfies the hypotheses of Corollary 1.4.7 and so our theorem applies to this situation. We will use this observation in Section 7.8.

6. Milnor excision for stable ∞-categories and equivalences of pro-∞-categories

The goal of this section is to define the notion of a Milnor square for stable ∞-categories. We prove results analogous to Milnor excision and pro-excision for localizing invariants in this setting. These results can be viewed as a generalization of the abstract excision results of [LT19] from ring spectra to arbitrary stable ∞-categories.

6.1. **The base change morphism.** We fix a commutative square of small stable ∞-categories

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{f} & \mathcal{B} \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ \mathcal{A}' & \xrightarrow{g} & \mathcal{B}'. \end{array} \tag{6.1.1}$$

Applying the cocompletion functor to the diagram we obtain a square

$$\begin{array}{ccc} \operatorname{Ind}(\mathcal{A}) & \underset{f_*}{\overset{f^*}{\rightleftarrows}} & \operatorname{Ind}(\mathcal{B}) \\ {\scriptstyle p_*}\uparrow\downarrow{\scriptstyle p^*} & & {\scriptstyle q_*}\uparrow\downarrow{\scriptstyle q^*} \\ \operatorname{Ind}(\mathcal{A}') & \underset{g_*}{\overset{g^*}{\rightleftarrows}} & \operatorname{Ind}(\mathcal{B}') \end{array} \tag{6.1.2}$$

In this context we have the base change transformation $\mathrm{E}x\colon f^*p_* \to q_*g^*$ of functors $\operatorname{Ind}(\mathcal{A}') \to \operatorname{Ind}(\mathcal{B})$ defined as the composition

$$f^*p_* \xrightarrow{\eta_q} q_*q^*f^*p_* \simeq q_*g^*p^*p_* \xrightarrow{q_*g^*\varepsilon_p} q_*g^*.$$

We say that the square (6.1.1) **satisfies base change** if $\mathrm{E}x$ is an equivalence. The easiest example of a square satisfying base change comes from taking $\operatorname{perf}(-)$ of a pushout diagram of $\mathcal{E}_\infty$-ring spectra. In fact, we will see that it is the only example coming from commutative case.

Any exact functor $\mathcal{B}' \xrightarrow{t} \mathcal{E}$ gives rise to a new commutative square

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{f} & \mathcal{B} \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle tq} \\ \mathcal{A}' & \xrightarrow{tg} & \mathcal{E}. \end{array}$$

We denote the base change transformation for the square by $\mathrm{E}x_{\mathcal{E}}\colon f^*p_* \to q_*t_*t^*g^*$.

Lemma 6.1.3. *For $\mathcal{E}$ and $\mathcal{B}' \xrightarrow{t} \mathcal{E}$ as above the two base change transformations satisfy* $\mathrm{E}x_{\mathcal{E}} \simeq q_*\eta_t(g^*) \circ \mathrm{E}x_{\mathcal{B}'}$.

Proof. This follows from commutativity of the diagram below.

$$\begin{array}{ccccccc} f^*p_* & \xrightarrow{\eta_q} & q_*q^*f^*p_* & \xrightarrow{\simeq} & q_*g^*p^*p_* & \xrightarrow{q_*g^*\varepsilon_p} & q_*g^* \\ \downarrow{=} & & \downarrow{\scriptstyle q_*\eta_t} & & \downarrow{\scriptstyle q_*\eta_t} & & \downarrow{\scriptstyle q_*\eta_t} \\ f^*p_* & \xrightarrow{\eta_{tq}} & q_*t_*t^*q^*f^*p_* & \xrightarrow{\simeq} & q_*t_*t^*g^*p^*p_* & \xrightarrow{q_*t_*t^*g^*\varepsilon_p} & q_*t_*t^*g^*. \end{array}$$

The rightmost and the middle squares are commutative because $q_*\eta_t$ is a natural transformation. The commutativity of the leftmost square is the standard property of adjunctions (see [ML13, VI.1, Theorem 1]). $\square$

Definition 6.1.4. We say that an exact functor $\mathcal{A} \xrightarrow{f} \mathcal{B}$ between stable ∞-categories is **dense** if its image generates $\mathcal{B}$ under finite limits, finite colimits, and taking retracts. Note that in this case $f_*\colon \operatorname{Ind}(\mathcal{B}) \to \operatorname{Ind}(\mathcal{A})$ is conservative and that dense functors are closed under composition.

Corollary 6.1.5. *Assume g and q are dense (see Definition 6.1.4) and (6.1.1) satisfies base change. Then any functor $\mathcal{B}' \xrightarrow{t} \mathcal{E}$ that induces a square satisfying base change is fully faithful.*

Proof. By Lemma 6.1.3 $q_*\eta_t(g^*)$ is an equivalence. Since g is dense and q_* is conservative η_t is also an equivalence. $\square$

6.1.6. We see from Corollary 6.1.5 that perf(−) of a commutative square of commutative rings satisfies base change if and only if it is a pushout square. Indeed, any such square

$$\begin{array}{ccc} \mathrm{perf}(\mathrm{R}) & \longrightarrow & \mathrm{perf}(\mathrm{S}) \\ \downarrow & & \downarrow \\ \mathrm{perf}(\mathrm{R}') & \longrightarrow & \mathrm{perf}(\mathrm{S}') \end{array}$$

is obtained from the square

$$\begin{array}{ccc} \mathrm{perf}(\mathrm{R}) & \longrightarrow & \mathrm{perf}(\mathrm{S}) \\ \downarrow & & \downarrow \\ \mathrm{perf}(\mathrm{R}') & \longrightarrow & \mathrm{perf}(\mathrm{R}' \otimes \mathrm{S}) \end{array}$$

by composing the lower horizontal and the right vertical functors with a functor $\mathrm{perf}(\mathrm{R}' \otimes_{\mathrm{R}} \mathrm{S}) \to \mathrm{perf}(\mathrm{S}')$. The latter functor is fully faithful by Corollary 6.1.5, so the map of rings $\mathrm{R}' \otimes_{\mathrm{R}} \mathrm{S} = \pi_0 \mathrm{E}nd_{\mathrm{perf}(\mathrm{R}' \otimes_{\mathrm{R}} \mathrm{S})}(\mathbf{1}) \to \pi_0 \mathrm{E}nd_{\mathrm{perf}(\mathrm{S}')}(\mathbf{1}) = \mathrm{S}'$ is an isomorphism.

6.1.7. Unlike the commutative rings case there turns out to be many more examples of squares satisfying base change in general. In particular, there are interesting commutative squares of **non-commutative** rings that give rise to squares satisfying base change.

Construction 6.1.8. Denote by $\mathcal{C}$ the lax pullback $\mathcal{A}' \times_{\mathcal{B}'}^{\rightarrow} \mathcal{B}$, i.e. the pullback of ∞-categories

$$\begin{array}{ccc} \mathcal{A}' \times_{\mathcal{B}'}^{\rightarrow} \mathcal{B} & \longrightarrow & \mathrm{Fun}(\Delta^1, \mathcal{B}') \\ \downarrow & & \downarrow \\ \mathcal{A}' \times \mathcal{B} & \longrightarrow & \mathcal{B}' \times \mathcal{B}'. \end{array}$$

The objects of the category are triples $(\mathrm{X}, \mathrm{Y}, f)$ with $\mathrm{X} \in \mathcal{A}'$, $\mathrm{Y} \in \mathcal{B}$ and $f \in \mathrm{Maps}_{\mathcal{B}'}(g(\mathrm{X}), q(\mathrm{Y}))$. By [Tam18, Proposition 10] this admits a semi-orthogonal decomposition

$$\mathcal{B} \underset{p_{\mathcal{B}}}{\overset{i_{\mathcal{B}}}{\rightleftarrows}} \mathcal{C} \underset{i_{\mathcal{A}'}}{\overset{p_{\mathcal{A}'}}{\rightleftarrows}} \mathcal{A}'$$

Here $i_{\mathcal{A}'}$ sends an object X of $\mathcal{A}'$ into $(\mathrm{X}, 0, 0)$, $p_{\mathcal{A}'}$ sends $(\mathrm{X}, \mathrm{Y}, f) \in \mathcal{C}$ into X, $i_{\mathcal{B}}$ sends $\mathrm{Y} \in \mathcal{B}$ into $\Sigma(0, \mathrm{Y}, 0)$, and $p_{\mathcal{B}}$ sends $(\mathrm{X}, \mathrm{Y}, f) \in \mathcal{C}$ into $\Omega\mathrm{Y}$. We chose $i_{\mathcal{B}}$ and $p_{\mathcal{B}}$ to be shifts of the obvious functors to get rid of shifts in further statements. We also have fiber sequences $i_{\mathcal{B}} p_{\mathcal{B}} \mathrm{X} \to \mathrm{X} \to i_{\mathcal{A}'} p_{\mathcal{A}'} \mathrm{X}$ functorial in X.

The commutative square (6.1.1) and the universal property of the lax pullback give rise to a functor $\mathcal{A} \xrightarrow{h} \mathcal{C}$ and equivalences $p_{\mathcal{A}'} \circ h \simeq p$ and $\Sigma p_{\mathcal{B}} \circ h \simeq f$. By definition, we have a functor $\mathcal{C} \to \mathrm{Fun}(\Delta^1, \mathcal{B}')$ which gives rise to a natural transformation

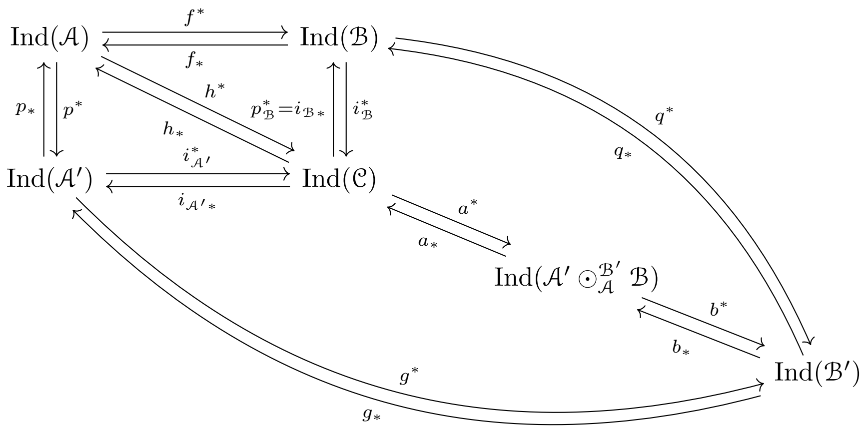
$$g \circ \Sigma p_{\mathcal{B}} \to q \circ p_{\mathcal{A}'}.$$

We denote the functor given by its cofiber by c. It sends an object $(\mathrm{M}, \mathrm{N}, f)$ of the lax pullback to the cofiber of f. Since $\mathrm{Cofib}(g \circ p \to q \circ f)$ is trivial, the composite functor $c \circ h$ is also trivial, so c factors as a composite $\mathcal{C} \xrightarrow{a} \mathcal{Q} \xrightarrow{b} \mathcal{B}'$, where a is the Verdier localization with respect to the image of h.

We will denote the ∞-category $\mathcal{Q}$ by $\mathcal{A}' \odot_{\mathcal{A}}^{\mathcal{B}'} \mathcal{B}$.

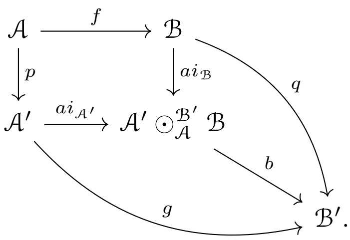
This can be wrapped up into the following diagram

(6.1.9)



Remark 6.1.10. If (6.1.1) is a commutative square of boundedly weighted stable ∞-categories then $\mathrm{Maps}_{\mathcal{C}}(i_{\mathcal{A}'}\mathrm{X}, i_{\mathcal{B}}\mathrm{Y}) = \mathrm{Maps}_{\mathcal{B}'}(g(\mathrm{X}), p(\mathrm{Y}))$ is connective for any $\mathrm{X} \in \mathcal{A}'^{\heartsuit}$ and $\mathrm{Y} \in \mathcal{B}^{\heartsuit}$, so $\mathcal{C}$ admits a weight structure such that all the functors in the diagram for semi-orthogonal decomposition are weight-exact (see Theorem 4.4.4). Hence, it follows from 2.4.7 that $\mathcal{A}' \odot_{\mathcal{A}}^{\mathcal{B}'} \mathcal{B}$ is canonically boundedly weighted in such a way that the functors $ai_{\mathcal{B}}$ and $ai_{\mathcal{A}}$ are weight-exact.

Proposition 6.1.11. *The following diagram is commutative:*



In other words, there is an equivalence $ai_{\mathcal{B}}f \simeq ai_{\mathcal{A}'}p$

Proof. Applying a to the fiber sequence $i_{\mathcal{B}}p_{\mathcal{B}}h \to h \to i_{\mathcal{A}'}p_{\mathcal{A}'}h$ we obtain a natural equivalence $ai_{\mathcal{B}}p_{\mathcal{B}}h \simeq \Omega ai_{\mathcal{A}'}p_{\mathcal{A}'}h$. There are equivalences $\Sigma p_{\mathcal{B}}h \simeq f$ and $p_{\mathcal{A}'}h \simeq p$ given by the fact that h was defined using the universal property of $\mathcal{C}$ and the functors p, f. Hence $ai_{\mathcal{B}}f \simeq ai_{\mathcal{A}'}p$.

By definition of $c \simeq ba$ we have $bai_{\mathcal{B}} \simeq \Sigma\,\mathrm{Fib}(0 \to q) \simeq q$ and $bai_{\mathcal{A}'} \simeq \mathrm{Fib}(g \to 0) \simeq g$. □

Definition 6.1.12. We say that a commutative square (6.1.1) is map-pullback if the functor $\mathcal{A} \to \mathcal{A}' \times_{\mathcal{B}'} \mathcal{B}$ is fully faithful. Note that any pullback square is map-pullback.

For a general commutative square (6.1.1) we denote by $\mathcal{A}' \times_{\mathcal{B}'}^{\mathcal{A}} \mathcal{B}$ the full subcategory of the pullback $\mathcal{A}' \times_{\mathcal{B}'} \mathcal{B}$ generated under finite limits and colimits and retracts by the image of $\mathcal{A}$. In other words this the minimal map-pullback square with a map from the original square.

6.1.13. Using adjunctions from the diagram (6.1.2) the condition on the square (6.1.1) being map-pullback can be reformulated into saying that the diagram

$$\begin{CD} \mathrm{X} @>{\eta_f}>> f_*f^*\mathrm{X} \\ @V{\eta_p}VV @VV{f_*\eta_q}V \\ p_*p^*\mathrm{X} @>{p_*\eta_g}>> p_*g_*g^*p^*\mathrm{X} \simeq f_*q_*q^*f^*\mathrm{X} \end{CD}$$

is pullback for all $\mathrm{X} \in \mathrm{Ind}(\mathcal{A})$.

Lemma 6.1.14. *(1) The functor h_* can be computed on the object* $(\mathrm{M}, \mathrm{N}, \alpha)$ *as the pullback* $p_*\mathrm{M} \times_{f_*q_*q^*\mathrm{N}} f_*\mathrm{N}$.

(2) If the square (6.1.1) is map-pullback then the functor h^* is fully faithful.

Proof. We have natural equivalences

$$\begin{gathered}\mathrm{Maps}_{\mathrm{Ind}(\mathcal{A})}(-, p_*\mathrm{M} \times_{f_*q_*q^*\mathrm{N}} f_*\mathrm{N}) \\ \simeq \\ \mathrm{Maps}_{\mathrm{Ind}(\mathcal{A})}(-, p_*\mathrm{M}) \times_{\mathrm{Maps}_{\mathrm{Ind}(\mathcal{A})}(-, f_*q_*q^*\mathrm{N})} \mathrm{Maps}_{\mathrm{Ind}(\mathcal{A})}(-, f_*\mathrm{N}) \\ \simeq \\ \mathrm{Maps}_{\mathrm{Ind}(\mathcal{A}')}(p^*(-), \mathrm{M}) \times_{\mathrm{Maps}_{\mathrm{Ind}(\mathcal{B}')}(q^*f^*(-), q^*\mathrm{N})} \mathrm{Maps}_{\mathrm{Ind}(\mathcal{B})}(f^*(-), \mathrm{N}) \\ \simeq \\ \mathrm{Maps}_{\mathrm{Ind}(\mathcal{C})}(h^*(-), (\mathrm{M}, \mathrm{N}, \alpha))\end{gathered}$$

proving the first claim. We also see from the equivalences that the unit morphism $\mathrm{X} \to h_*h^*\mathrm{X}$ can be identified with $\mathrm{X} \overset{(\eta_p, \eta_f)}{\to} p_*p^*\mathrm{X} \times_{f_*q_*q^*f^*\mathrm{X}} f_*f^*\mathrm{X}$. This map is an equivalence if the square was map-pullback by 6.1.13, thus h^* is fully faithful. □

Theorem 6.1.15. *If the square (6.1.1) is map-pullback then the base change map for the commutative square in Lemma 6.1.11 is an equivalence on objects from the image of p^*. In other words, the map*

$$f^*p_*p^* \to i_{\mathcal{B}*}a_*a^*i^*_{\mathcal{A}'}p^*$$

is an equivalence. In particular, the square satisfies base change whenever p is dense.

Proof. It follows from the definition of $\mathcal{A}' \odot^{\mathcal{B}'}_{\mathcal{A}} \mathcal{B}$ and Lemma 6.1.14(2) that the counit of h and the unit of a give rise to a fiber sequence

$$h^*h_*\mathrm{W} \to \mathrm{W} \to a_*a^*\mathrm{W}$$

functorial in $\mathrm{W} \in \mathcal{C}$ (see 4.4.3). This gives a fiber sequence

$$i_{\mathcal{B}*}h^*h_*i^*_{\mathcal{B}}f^*\mathrm{X} \overset{i_{\mathcal{B}*}\varepsilon_h}{\to} i_{\mathcal{B}*}i^*_{\mathcal{B}}f^*\mathrm{X} \to i_{\mathcal{B}*}a_*a^*i^*_{\mathcal{B}}f^*\mathrm{X}$$

functorial in $\mathrm{X} \in \mathcal{A}$. In the notation of the diagram (6.1.9) $i_{\mathcal{B}*} = p^*_{\mathcal{B}}$, so by definition of h and by Lemma 6.1.14(1) there is an equivalence

$$\alpha \colon f^*\,\mathrm{Fib}(f_*f^*\mathrm{X} \to f_*q_*q^*f^*\mathrm{X}) \simeq i_{\mathcal{B}*}h^*\,\mathrm{Fib}(f_*f^*\mathrm{X} \to f_*q_*q^*f^*\mathrm{X}) \simeq i_{\mathcal{B}*}h^*h_*i^*_{\mathcal{B}}f^*\mathrm{X}$$

and the map $i_{\mathcal{B}*}\varepsilon_h$ corresponds to the map

$$f^*\,\mathrm{Fib}(f_*f^*\mathrm{X} \to f_*q_*q^*f^*\mathrm{X}) \overset{f^*i_{\mathrm{Fib}}}{\to} f^*f_*f^*\mathrm{X} \overset{\varepsilon_f}{\to} f^*\mathrm{X} \overset{\eta_{i_{\mathcal{B}}}}{\to} i_{\mathcal{B}*}i^*_{\mathcal{B}}f^*\mathrm{X}$$

under this equivalence. Since the square (6.1.1) is a map-pullback, the natural map $\mathrm{Fib}(\mathrm{X} \to p_*p^*\mathrm{X}) \overset{\beta}{\to} f_*\,\mathrm{Fib}(f^*\mathrm{X} \to q_*q^*f^*\mathrm{X})$ is an equivalence (see 6.1.13).

Now consider the diagram

(6.1.16)
$$\begin{array}{ccccc}
\mathrm{Fib}(f^*\mathrm{X} \to f^*p_*p^*\mathrm{X}) & \xrightarrow{i_{\mathrm{Fib}}} & f^*\mathrm{X} & \xrightarrow{\eta_{a i_{\mathcal{B}}}} & f^*p_*p^*\mathrm{X} \\
\downarrow \beta & & \downarrow = & & \downarrow \mathrm{E}x(p^*\mathrm{X}) \\
f^*\,\mathrm{Fib}(f_*f^*\mathrm{X} \to f_*q_*q^*f^*\mathrm{X}) & \xrightarrow{\varepsilon_f \circ f^*i_{\mathrm{Fib}}} & f^*\mathrm{X} & & i_{\mathcal{B}*}a_*a^*i^*_{\mathcal{A}'}p^*\mathrm{X} \\
\downarrow \alpha & & \downarrow \eta_{i_{\mathcal{B}}} & & \simeq\uparrow \\
i_{\mathcal{B}*}h^*h_*i^*_{\mathcal{B}}f^*\mathrm{X} & \xrightarrow{i_{\mathcal{B}*}\varepsilon_h} & i_{\mathcal{B}*}i^*_{\mathcal{B}}f^*\mathrm{X} & \xrightarrow{i_{\mathcal{B}*}\eta_a} & i_{\mathcal{B}*}a_*a^*i^*_{\mathcal{B}}f^*\mathrm{X}.
\end{array}$$

Its rows are fiber sequences and the maps α, β are equivalences. The map $\eta_{i_{\mathcal{B}}}$ is also an equivalence since $i_{\mathcal{B}}$ is fully faithful. Hence to show that the map $\mathrm{E}x(p^*\mathrm{X})$ is an equivalence it suffices to check that the diagram commutes.

We have already shown that the lower-left square commutes. By definition of β the diagram

$$\begin{array}{ccccc} \mathrm{Fib}(f^*\mathrm{X} \to f^*p_*p^*\mathrm{X}) & \xrightarrow{\beta} & f^*\,\mathrm{Fib}(f_*f^*\mathrm{X} \to f_*q_*q^*f^*\mathrm{X}) & & \\ \downarrow{\scriptstyle i_{\mathrm{Fib}}} & & \downarrow{\scriptstyle f^*i_{\mathrm{Fib}}} & & \\ f^*\mathrm{X} & \xrightarrow{f^*\eta_f} & f^*f_*f^*\mathrm{X} & \xrightarrow{\varepsilon_f} & f^*\mathrm{X} \end{array}$$

commmutes. The unit-counit equalities show that the composition of the bottom horizontal maps is homotopic to identity. Hence $\varepsilon_f \circ f^*i_{\mathrm{Fib}} \circ \beta \simeq i_{\mathrm{Fib}}$, i.e. the upper-left square in the diagram (6.1.16) commutes.

It suffices to show that the right part of the diagram commutes. To do that consider the diagram

$$\begin{array}{ccc} f^*\mathrm{X} & \xrightarrow{f^*\eta_p} & f^*p_*p^*\mathrm{X} \\ \downarrow{\scriptstyle \eta_{ai_{\mathcal{B}}}} & & \downarrow{\scriptstyle \eta_{ai_{\mathcal{B}}}} \\ i_{\mathcal{B}*}a_*a^*i^*_{\mathcal{B}}f^*\mathrm{X} & \xrightarrow{i_{\mathcal{B}*}a_*a^*i^*_{\mathcal{B}}\eta_p} & i_{\mathcal{B}*}a_*a^*i^*_{\mathcal{B}}f^*p_*p^*\mathrm{X} \\ \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} \\ i_{\mathcal{B}*}a_*a^*i^*_{\mathcal{A}'}p^*\mathrm{X} & \xrightarrow{i_{\mathcal{B}*}a_*a^*i^*_{\mathcal{A}'}\eta_p} & i_{\mathcal{B}*}a_*a^*i^*_{\mathcal{A}'}p^*p_*p^*\mathrm{X} \\ \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle i_{\mathcal{B}*}a_*a^*i^*_{\mathcal{A}'}\varepsilon_p} \\ i_{\mathcal{B}*}a_*a^*i^*_{\mathcal{A}'}p^*\mathrm{X} & \xrightarrow{=} & i_{\mathcal{B}*}a_*a^*i^*_{\mathcal{A}'}p^*\mathrm{X} \end{array}$$

Its top square is commutative by naturality of $\eta_{ai_{\mathcal{B}}}$, the middle square – by naturality of the equivalence $i^*_{\mathcal{B}}f^* \simeq i^*_{\mathcal{A}'}p^*$, and the bottom square just illustrates one of the unit-counit equalities. Reading off the equality for the bounding paths in the diagram we get commutativity of the right part of (6.1.16).

Since $\mathrm{E}x(p^*(-))$ is a natural transformation of functors commuting with colimits, the fact that $\mathrm{E}x(p^*(\mathrm{X}))$ is an equivalence for all $\mathrm{X} \in \mathcal{A}$ implies that it is an equivalence in general. □

Remark 6.1.17. Theorem 6.1.11 can be seen as a generalization of Proposition 1.13 of [LT19].

6.2. **Milnor excision results.**

Definition 6.2.1. Assume all the functors in the square (6.1.1) are dense. Then it is called a **Milnor square** if the following conditions hold

(1) it is a map-pullback square of ∞-categories,

(2) the base change transformation is an equivalence.

Lemma 6.2.2. *If p is dense then so is $\mathcal{B} \overset{ai_{\mathcal{B}}}{\to} \mathcal{A}' \odot^{\mathcal{B}'}_{\mathcal{A}} \mathcal{B}$. If f is dense then so is $\mathcal{A}' \overset{ai_{\mathcal{A}'}}{\to} \mathcal{A}' \odot^{\mathcal{B}'}_{\mathcal{A}} \mathcal{B}$.*

Proof. We only prove the first statement and the proof of the second statement is identical. Since p is dense, the category $\mathcal{C}$ is generated by the images of the functors $i_{\mathcal{B}}$ and $i_{\mathcal{A}'}p$. Being a localization functor, $\mathcal{C} \to \mathcal{A}' \odot^{\mathcal{B}'}_{\mathcal{A}} \mathcal{B}$ is surjective on objects, hence $\mathcal{A}' \odot^{\mathcal{B}'}_{\mathcal{A}} \mathcal{B}$ is also generated by the images of the functors $ai_{\mathcal{B}}$ and $ai_{\mathcal{A}'}p$. By Proposition 6.1.11 $ai_{\mathcal{A}'}p = ai_{\mathcal{B}}f$, so any object of $\mathcal{A}' \odot^{\mathcal{B}'}_{\mathcal{A}} \mathcal{B}$ is in the image of $ai_{\mathcal{B}}$. □

Proposition 6.2.3. *The second condition in the definition is equivalent to saying that the functor $b\colon \mathcal{A}' \odot^{\mathcal{B}'}_{\mathcal{A}} \mathcal{B} \to \mathcal{B}'$ is an equivalence up to idempotent completion.*

Proof. It follows from Theorem 6.1.15 that the commutative square involving $\mathcal{A}' \odot^{\mathcal{B}'}_{\mathcal{A}} \mathcal{B}$ satisfies base change. Hence the only thing left to prove is that b induces an equivalence on idempotent completions for a Milnor square. It follows from Lemma 6.2.2 and Corollary 6.1.5 that b is fully faithful. The functor b is also dense since q is. Therefore, b must be an equivalence up to idempotent completion. □

Our goal is to prove the following

Theorem 6.2.4 (Milnor excision for stable ∞-categories)**.** *Any localizing invariant sends a Milnor square of stable ∞-categories into a pullback square.*

Proposition 6.2.3 implies that it is safe to assume $\mathcal{B}' = \mathcal{A}' \odot_{\mathcal{A}}^{\mathcal{B}'} \mathcal{B}$. Hence Theorem 6.2.4 follows from the next proposition.

Proposition 6.2.5. *If the square (6.1.1) is map-pullback then any localizing invariant* E *sends the commutative square*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{f} & \mathcal{B} \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle ai_{\mathcal{B}}} \\ \mathcal{A}' & \xrightarrow{ai_{\mathcal{A}'}} & \mathcal{A}' \odot_{\mathcal{A}}^{\mathcal{B}'} \mathcal{B} \end{array}$$

into a pullback square of spectra.

Proof. We consider the diagram

$$\begin{array}{ccccc} \mathrm{E}(\mathcal{A}) & \xrightarrow{h} & \mathrm{E}(\mathcal{C}) & \xrightarrow{a} & \mathrm{E}(\mathcal{A}' \odot_{\mathcal{A}}^{\mathcal{B}'} \mathcal{B}) \\ \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle (p_{\mathcal{A}'}, \Sigma p_{\mathcal{B}})} & & \downarrow{\scriptstyle =} \\ \mathrm{E}(\mathcal{A}) & \xrightarrow{(p,f)} & \mathrm{E}(\mathcal{A}') \times \mathrm{E}(\mathcal{B}) & \xrightarrow{ai_{\mathcal{A}'} - ai_{\mathcal{B}}} & \mathrm{E}(\mathcal{A}' \odot_{\mathcal{A}}^{\mathcal{B}'} \mathcal{B}). \end{array}$$

It follows from the definition of $\mathcal{A}' \odot_{\mathcal{A}}^{\mathcal{B}'} \mathcal{B}$ and Lemma 6.1.14(2) that the top row in the diagram is a fiber sequence. The left square in the diagram is commutative by definition of h. The additivity theorem applied to the fiber sequence $a \to ai_{\mathcal{A}'}p_{\mathcal{A}'} \to \Sigma ai_{\mathcal{B}}p_{\mathcal{B}}$ yields that the right square is also commutative. Since $\mathcal{C}$ admits a semiorthogonal decomposition into $\mathcal{A}'$ and $\mathcal{B}$, the map $\mathrm{E}(\mathcal{C}) \to \mathrm{E}(\mathcal{A}' \times \mathcal{B})$ is a weak equivalence. Hence the bottom sequence is a fiber sequence and the square in question is a pullback square. □

6.3. **Pro-∞-categories.** For a small ∞-category $\mathcal{A}$ the category of pro-objects $\mathrm{Pro}(\mathcal{A})$ is the full subcategory of $\mathrm{Fun}(\mathcal{A}, \mathrm{Spc})^{op}$ spanned by small cofiltered limits of representables. This is also equivalent to $\mathrm{Ind}(\mathcal{A}^{op})^{op}$. There is a fully faithful embedding $\mathcal{A} \to \mathrm{Pro}(\mathcal{A})$ and any cofiltered system $\{\mathrm{X}_i\}_{i \in \mathrm{I}}$ of objects of $\mathcal{A}$ gives rise to a pro-object which we denote by "lim"X_i. Any morphism between such pro-objects "lim"X_i and "lim"Y_i in $\mathrm{Pro}(\mathcal{A})$ is equivalent to a morphism that comes from a natural transformation of cofiltered systems. More precisely, there is a canonical equivalence $\mathrm{Maps}_{\mathrm{Pro}(\mathcal{A})}(\text{“lim”}\mathrm{X}_i, \text{“lim”}\mathrm{Y}_j) \simeq \lim_j \mathrm{colim}_i \mathrm{Maps}_{\mathrm{Pro}(\mathcal{A})}(\mathrm{X}_i, \mathrm{Y}_j)$ (see Section 5.1 of [BHH17] or Remark A.8.1.5 of [Lur18]).

Composing with the functor $\mathcal{A} \to \mathrm{Pro}(\mathcal{A})$ gives an equivalence of ∞-categories

$$\mathrm{Fun}'_{\mathcal{A}}(\mathrm{Pro}(\mathcal{A}), \mathcal{B}) \to \mathrm{Fun}(\mathcal{A}, \mathcal{B})$$

for any ∞-category $\mathcal{B}$ that admits small cofiltered limits where the left hand-side is the subcategory of $\mathrm{Fun}_{\mathcal{A}}(\mathrm{Pro}(\mathcal{A}), \mathcal{B})$ spanned by small cofiltered limits-preserving functors (see Theorem 3.2.19 of [BHH17] or Proposition A.8.1.6 of [Lur18]). In particular, for any functor $\mathcal{A} \xrightarrow{f} \mathcal{B}$ of small ∞-categories there is functor $\mathrm{Pro}(\mathcal{A}) \xrightarrow{\mathrm{Pro}(f)} \mathrm{Pro}(\mathcal{B})$ that preserves small cofiltered limits such that the diagram

$$\begin{array}{ccc} \mathcal{A} & \longrightarrow & \mathcal{B} \\ \downarrow & & \downarrow \\ \mathrm{Pro}(\mathcal{A}) & \longrightarrow & \mathrm{Pro}(\mathcal{B}) \end{array}$$

is commutative. Furthermore, if $\mathcal{A}$ and $\mathcal{B}$ admit finite limits and f commutes with them, then the functor $\mathrm{Pro}(f)$ admits a left adjoint (see Proposition 5.3.5.13 of [Lur17b] or Example A.8.1.8 of [Lur18]). If $\mathcal{A}$ is stable, $\mathrm{Pro}(\mathcal{A})$ is also stable by Proposition 1.1.3.6 of [Lur17a].

The following result will also be very useful to us.

Lemma 6.3.1. *The functor* $\mathrm{Fun}(\mathrm{I}, \mathcal{A}) \to \mathrm{Pro}(\mathcal{A})$ *that sends a filtered system* $\{\mathrm{X}_i\}_{i\in \mathrm{I}}$ *to* "$\lim$"X_i *commutes with all finite limits and finite colimits that exist in* $\mathrm{Fun}(\mathrm{I}, \mathcal{A})$.

Proof. This is dual to [Lur17b, Proposition 5.3.4.14]. □

Now let $\mathcal{A}$ be either Spt, $\mathrm{Alg}^{\mathrm{cn}}_{\mathbb{E}_1}$, $\mathbf{Mod}_{\mathrm{R}}$ for some connective $\mathbb{E}_1$-ring R or $\mathbf{WCat}^{\mathrm{st},b}_{\infty}$, $\mathbf{Cat}^{\mathrm{Kar}}_{\infty}$. In this case we have a family of functors $\tau_{\leqslant n} \colon \mathcal{A} \to \mathcal{A}$ given by Postnikov truncations (see Remark 3.2.12). They induce functors $\mathrm{Pro}(\mathcal{A}) \to \mathrm{Pro}(\mathcal{A})$ which we refer to using the same notation. By definition they commute with cofiltered limits.

Definition 6.3.2. We say that a morphism $f \in \mathrm{Pro}(\mathcal{A})$ is a **weak equivalence** if $\tau_{\leqslant n} f$ is an equivalence for all n.

We will say that a cone X is a weak limit of a diagram A_n in $\mathrm{Pro}(\mathcal{A})$ if the natural map $\mathrm{X} \to \lim \mathrm{A}_n$ is a weak equivalence.

Denote by $\mathcal{A}^+$ the subcategory of bounded above objects of $\mathcal{A}$. The induced functor $\mathrm{Pro}(\mathcal{A}^+) \to \mathrm{Pro}(\mathcal{A})$ is fully faithful and admits a left adjoint which we denote by $\tau_{<\infty}$. The functor can be explicitly described as $\lim_n \tau_{\leqslant n}$ (see [KST18, Lemma 2.6]). This implies that the map $\mathrm{X} \to \tau_{<\infty}\mathrm{X}$ is a weak equivalence for any X. Thus, a map $\mathrm{X} \xrightarrow{f} \mathrm{Y}$ is a weak equivalence if and only if $\tau_{<\infty} f$ is an equivalence (see also [KST18, Lemma 2.7]).

Definition 6.3.3. We call an object $\mathrm{X} \in \mathrm{Pro}(\mathcal{A})$ **pro-truncated** if $\mathrm{X} \to \tau_{<\infty}\mathrm{X}$ is an equivalence. By the discussion above, this is equivalent to saying that X can be represented as a cofiltered limit of bounded objects of $\mathcal{A}$.

Lemma 6.3.4. *Assume* $\mathcal{A} = \mathbf{Mod}_{\mathrm{R}}$ *for some connective* $\mathbb{E}_1$*-ring spectrum* R. *The functor* $\tau_{<\infty}$ *commutes with finite limits. A finite limit of weak equivalences in* $\mathcal{A}$ *is a weak equivalence.*

Proof. Let $\{\mathrm{X}_i\}$ be a finite diagram in $\mathrm{Pro}(\mathcal{A})$. We have a fiber sequence of underlying pro-spectra

$$\lim_i \tau_{\geqslant k+1}\mathrm{X}_i \to \lim_i \mathrm{X}_i \to \lim_i \tau_{\leqslant k}\mathrm{X}_i.$$

Since the limit diagram is finite, there exists l depending only on the diagram such that the first term in the fiber sequence above is $(k-l)$-connective. Consequently,

$$\tau_{\leqslant n} \lim_i \mathrm{X}_i \to \tau_{\leqslant n} \lim_i \tau_{\leqslant k}\mathrm{X}_i \text{ and}$$

$$\tau_{\leqslant n} \lim_i \tau_{<\infty}\mathrm{X}_i \to \tau_{\leqslant n} \lim_i \tau_{\leqslant k}\mathrm{X}_i$$

are equivalences for $k > n + l$. Thus

$$\tau_{\leqslant n} \lim_i \mathrm{X}_i \to \tau_{\leqslant n} \lim_i \tau_{<\infty}\mathrm{X}_i$$

is an equivalence for any n. By [LT19, Lemma 2.8] weak equivalences in $\mathcal{A}$ can be checked on the underlying pro-spectra, so $\tau_{<\infty}$ commutes with finite limits.

Finally for any finite diagram of maps that are weak equivalences $\{f_n\}$ the morphism

$$\lim_n \tau_{<\infty} f_n \simeq \tau_{<\infty} \lim_n f_n$$

is an equivalence. □

6.3.5. *Equivalences of pro-∞-categories.* We now consider the case of pro-objects in $\mathbf{Cat}_\infty^{\mathrm{Kar}}$, i.e. the ∞-category of small idempotent complete additive ∞-categories. The goal of this subsection is to give a criterion for "lim"f_λ to be an equivalence for a morphism of cofiltered systems of additive ∞-categories $\{\mathcal{A}_\lambda\} \overset{\{f_\lambda\}}{\to} \{\mathcal{B}_\lambda\}$.

Definition 6.3.6. Let $\{\mathcal{A}_\lambda\}$, $\{\mathcal{B}_\lambda\}$ be cofiltered systems of idempotent complete additive ∞-categories.

- An **object** of $\{\mathcal{A}_\lambda\}$ is a morphism from the constant family $\{pt\}$ to $\{\mathcal{A}_\lambda\}$, where pt is the one-point space. For an object $\mathrm{X}\colon \{pt\} \to \{\mathcal{A}_\lambda\}$ we denote the value of the λ-th functor by X_λ.
- For an object X of $\{\mathcal{A}_\lambda\}$ and a morphism $\{\mathcal{A}_\lambda\} \overset{\{f_\lambda\}}{\to} \{\mathcal{B}_\lambda\}$, the **image** of X is the composite functor. We denote it by $f(\mathrm{X})$ when there is no confusion.
- We say that $\{\mathcal{A}_\lambda\}$ **has enough objects** if there exists a set S and a morphism from the constant system $\{\mathrm{S}\}$ to $\{\mathcal{A}_\lambda\}$ which generates the image under taking retracts and sums levelwisely.
- We say that $\{\mathcal{A}_\lambda\}$ is **bounded** if there exists an integer n such that for all $i > n$ and indices λ $\pi_i \operatorname{Maps}_{\mathcal{A}_\lambda}(\mathrm{X}, \mathrm{Y}) = 0$ for all $\mathrm{X}, \mathrm{Y} \in \mathcal{A}_\lambda$.

Any two objects X, Y of a cofiltered system of ∞-categories $\{\mathcal{A}_\lambda\}$ give rise to a cofiltered system of spaces $\{\operatorname{Maps}_{\mathcal{A}_\lambda}(\mathrm{X}_\lambda, \mathrm{Y}_\lambda)\}$. This is a pro-invariant, namely, any pro-equivalence of cofiltered systems $\{\mathcal{A}_\lambda\} \overset{\{f_\lambda\}}{\to} \{\mathcal{B}_\lambda\}$ with a fixed pair of objects induces an equivalence

$$\text{“}\lim\text{”} \operatorname{Maps}_{\mathcal{A}_\lambda}(\mathrm{X}_\lambda, \mathrm{Y}_\lambda) \to \text{“}\lim\text{”} \operatorname{Maps}_{\mathcal{B}_\lambda}(f(\mathrm{X})_\lambda, f(\mathrm{Y})_\lambda)$$

since the mapping spaces are functorial. Given a small set S of objects of $\{\mathcal{A}_\lambda\}$ we can as well consider the cofiltered family of $\mathrm{S} \times \mathrm{S}$-indexed spaces $\{\operatorname{Maps}_{\mathcal{A}_\lambda}(s_\lambda, s'_\lambda) | s, s' \in \mathrm{S}\}$. Similarly, any pro-equivalence of cofiltered systems of ∞-categories over S $\{\mathcal{A}_\lambda\} \overset{\{f_\lambda\}}{\to} \{\mathcal{B}_\lambda\}$ induces an equivalence

$$\text{“}\lim\text{”} \operatorname{Maps}_{\mathcal{A}_\lambda}(s_\lambda, s'_\lambda) \to \text{“}\lim\text{”} \operatorname{Maps}_{\mathcal{B}_\lambda}(f(s_\lambda), f(s'_\lambda))$$

in $\operatorname{Pro}(\mathrm{Spc}^{\mathrm{S}})$.

It turns out that "lim" $\operatorname{Maps}_{\mathcal{A}_\lambda}(\mathrm{X}_\lambda, \mathrm{Y}_\lambda)$ is a complete pro-invariant on bounded above pro-systems of idempotent complete additive ∞-categories that have enough objects.

Theorem 6.3.7. *Let $\{\mathcal{A}_\lambda\} \overset{\{f_\lambda\}}{\to} \{\mathcal{B}_\lambda\}$ be a morphism of bounded above cofiltered systems of idempotent complete additive ∞-categories which is levelwise additive and essentially surjective. Assume $\{\mathcal{A}_\lambda\}$ has enough objects given by a set* S*. If the map*

$$\text{“}\lim\text{”} \operatorname{Maps}_{\mathcal{A}_\lambda}(s_\lambda, s'_\lambda) \to \text{“}\lim\text{”} \operatorname{Maps}_{\mathcal{B}_\lambda}(f(s_\lambda), f(s'_\lambda))$$

is an equivalence of pro-$\mathrm{S} \times \mathrm{S}$*-indexed spaces, then* "lim"f_λ *is also an equivalence.*

Proof. **Easy case.** We start with the case $\mathrm{S} = \{*\}$ Recall from Theorem 5.3.1 that we have a fully faithful functor

$$\Theta : \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Spt}^{\mathrm{cn}}) \longrightarrow \mathbf{Cat}^{\mathrm{Kar}}_{\infty, \mathbf{Proj}^{\mathrm{fg}}_{\mathbb{S}}/}$$

whose image consists of functors $\mathbf{Proj}^{\mathrm{fg}}_{\mathbb{S}} \overset{\mathrm{F}}{\to} \mathcal{A}$ with $\mathcal{A}$ generated by $\mathrm{F}(\mathbb{S})$ under finite sums and retracts. This induces a fully faithful functor

$$\operatorname{Pro}(\Theta) : \operatorname{Pro}(\mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Spt}^{\mathrm{cn}})) \longrightarrow \operatorname{Pro}(\mathbf{Cat}^{\mathrm{Kar}}_{\infty, \mathbf{Proj}^{\mathrm{fg}}_{\mathbb{S}}/})$$

whose image consists of "lim"$\mathcal{A}_\lambda$ for those cofiltered systems $\{\mathcal{A}_\lambda\}$ that have enough objects given by the set $\mathrm{S} = \{*\}$. Moreover, "lim"$\mathcal{A}_\lambda$ is equivalent to $\operatorname{Pro}(\Theta)(\text{“}\lim\text{”} \operatorname{End}_{\mathcal{A}_\lambda}(\mathrm{X}_\lambda))$.

Now for a morphism $\{\mathcal{A}_\lambda\} \overset{\{f_\lambda\}}{\to} \{\mathcal{B}_\lambda\}$ as in the statement, the map

$$g : \text{“}\lim\text{”}\,\mathrm{End}_{\mathcal{A}_\lambda}(\mathrm{X}_\lambda) \to \text{“}\lim\text{”}\,\mathrm{End}_{\mathcal{B}_\lambda}(f_\lambda \mathrm{X}_\lambda)$$

is an equivalence of pro-spectra by assumption. These are also bounded from above, so by [LT19, Lemma 2.28(ii)] this map is also an equivalence as a map of pro-$\mathbb{E}_1$-ring spectra. Now by the observation above, $\mathrm{Pro}(\Theta)(g)$ is equivalent to “$\lim$”f_λ, so the latter is also an equivalence as claimed.

General case. The proof is similar to the proof of Theorem 5.3.3. We construct a diagram in $\mathrm{Fun}(\mathrm{I}, \mathbf{Cat}_\infty^{\mathrm{Kar}})$

$$\begin{array}{ccccc} \{\mathcal{A}_\lambda\} & \longrightarrow & \{\mathcal{A}_\lambda^{\mathrm{big}}\} & \longrightarrow & \{\mathcal{A}_\lambda^{\mathrm{quot}}\} \\ \downarrow & & \downarrow & & \downarrow \\ \{\mathcal{B}_\lambda\} & \longrightarrow & \{\mathcal{B}_\lambda^{\mathrm{big}}\} & \longrightarrow & \{\mathcal{B}_\lambda^{\mathrm{quot}}\} \end{array} \tag{6.3.8}$$

such that the following properties hold:

- the rows are levelwise fiber sequences,
- the middle and the right vertical functors are levelwise essentially surjective up to retracts,
- $\{\mathcal{A}_\lambda^{\mathrm{big}}\}$, $\{\mathcal{A}_\lambda^{\mathrm{quot}}\}$, $\{\mathcal{B}_\lambda^{\mathrm{big}}\}$, $\{\mathcal{B}_\lambda^{\mathrm{quot}}\}$ are all bounded and have enough objects given by a one-element set.

Given an additive ∞-category $\mathcal{A}$, a set S, and a map $p : \mathrm{S} \to \mathcal{A}$ whose image generates $\mathcal{A}$ define $\mathcal{A}^{\mathrm{big}}$ to be the minimal full subcategory of $\mathrm{P}_\Sigma(\mathcal{A})$ containing the object $\mathrm{G} = \bigoplus_{s\in \mathrm{S}} p(s)$ and closed under taking direct sums and summands. It contains $\mathcal{A}$ as a full subcategory and is itself equivalent to $\mathbf{Proj}^{\mathrm{fg}}_{\mathrm{End(G)}}$. Define $\mathcal{A}^{\mathrm{quot}}$ to be the homotopy category of the idempotent completion of the additive quotient[9] $\frac{\mathcal{A}^{\mathrm{big}}}{\mathcal{A}}$ (see Section 2.2). As an idempotent additive ∞-category it is generated by the image of G, so it is equivalent to $\mathbf{Proj}^{\mathrm{fg}}_{\mathrm{Q}}$ for a discrete $\mathcal{E}_1$-ring spectrum Q. The constructions are functorial so we obtain the diagram (6.3.8) by applying them levelwisely to $\{\mathcal{A}_\lambda\}$ and $\{\mathcal{B}_\lambda\}$. The second and the last required properties hold by construction.

The fiber of a functor $\mathcal{A} \to \mathcal{B}$ in $\mathbf{Cat}^{\mathrm{Kar}}$ is computed as the full subcategory of $\mathcal{A}$ consisting of those objects that map to zero object. The formula Corollary 2.2.4(2) implies that if the functor $\mathcal{A}^{\mathrm{big}} \to \mathcal{A}^{\mathrm{big}}/\mathcal{A}$ maps $\mathrm{X} \in \mathcal{A}^{\mathrm{big}}$ to a zero object, then id_X must factor through an object of $\mathcal{A}$. Since $\mathcal{A}$ is idempotent complete, this means that X also belongs to $\mathcal{A}$. Now the functor

$$\mathcal{A}^{\mathrm{big}}/\mathcal{A} \to h(\mathcal{A}^{\mathrm{big}}/\mathcal{A}) = \mathcal{A}^{\mathrm{quot}}$$

is conservative, so we actually have an equivalence

$$\mathcal{A} \simeq \mathrm{Fib}(\mathcal{A}^{\mathrm{big}} \to \mathcal{A}^{\mathrm{quot}}).$$

In particular the diagram we constructed satisfies the first required property.

By Lemma 6.3.1 taking formal limits of cofiltered diagrams commutes with finite limits, therefore to prove the claim it suffices to show that the middle and the right vertical maps in the diagram are pro-equivalences. By the easy case proved above it suffices to show that the following maps of pro-spectra are equivalences:

$$\text{“}\lim\text{”}\,\mathrm{Maps}_{\mathrm{P}_\Sigma(\mathcal{A}_\lambda)}(\mathrm{G}_{\mathcal{A}_\lambda}, \mathrm{G}_{\mathcal{A}_\lambda}) \to \text{“}\lim\text{”}\,\mathrm{Maps}_{\mathrm{P}_\Sigma(\mathcal{B}_\lambda)}(\mathrm{G}_{\mathcal{B}_\lambda}, \mathrm{G}_{\mathcal{B}_\lambda}),$$

$$\text{“}\lim\text{”}\,\mathrm{Maps}_{\mathcal{A}_\lambda^{\mathrm{quot}}}(\mathrm{G}_{\mathcal{A}_\lambda}, \mathrm{G}_{\mathcal{A}_\lambda}) \to \text{“}\lim\text{”}\,\mathrm{Maps}_{\mathcal{B}_\lambda^{\mathrm{quot}}}(\mathrm{G}_{\mathcal{B}_\lambda}, \mathrm{G}_{\mathcal{B}_\lambda}).$$

[9] we only need to consider the homotopy category here to ensure boundedness of the quotient.

The first one can be equivalently rewritten as

$$\text{“}\lim\text{”} \bigoplus_{s,s' \in \mathrm{S}} \mathrm{Maps}_{\mathcal{A}_\lambda}(s_\lambda, s'_\lambda) \to \text{“}\lim\text{”} \bigoplus_{s,s' \in \mathrm{S}} \mathrm{Maps}_{\mathcal{B}_\lambda}(f_\lambda s_\lambda, f_\lambda s'_\lambda)$$

which is an equivalence as it is the image of the map of pro-$\mathrm{S}\times\mathrm{S}$-indexed spaces in the statement under the direct sum functor

$$\mathrm{Pro}(\mathrm{Fun}(\mathrm{S}\times\mathrm{S}, \mathrm{Spc})) \to \mathrm{Pro}(\mathrm{Spc}).$$

Using Corollary 2.2.4(2) and Lemma 6.3.1 we can write the second map as the induced map on the cokernels of the horizontal maps in the following diagram of groups

$$\begin{array}{ccc} \text{“}\lim\text{”} \bigoplus_{s,q,t\in\mathrm{S}} \pi_0 \mathrm{Maps}_{\mathcal{A}_\lambda}(q_\lambda, t_\lambda) \otimes \pi_0 \mathrm{Maps}_{\mathcal{A}_\lambda}(s_\lambda, q_\lambda) & \longrightarrow & \text{“}\lim\text{”} \bigoplus_{s,t\in\mathrm{S}} \pi_0 \mathrm{Maps}_{\mathcal{A}_\lambda}(s_\lambda, t_\lambda) \\ \downarrow & & \downarrow \\ \text{“}\lim\text{”} \bigoplus_{s,q,t\in\mathrm{S}} \pi_0 \mathrm{Maps}_{\mathcal{B}_\lambda}(f_\lambda q_\lambda, f_\lambda t_\lambda) \otimes \pi_0 \mathrm{Maps}_{\mathcal{B}_\lambda}(f_\lambda s_\lambda, f_\lambda q_\lambda) & \longrightarrow & \text{“}\lim\text{”} \bigoplus_{s,t\in\mathrm{S}} \pi_0 \mathrm{Maps}_{\mathcal{B}_\lambda}(f_\lambda s_\lambda, f_\lambda t_\lambda). \end{array}$$

Both vertical maps are functorial images of the map of pro-$\mathrm{S}\times\mathrm{S}$-indexed spaces in the statement that is assumed to be an equivalence, so the map on the cokernels is also a pro-equivalence. □

Corollary 6.3.9. *Let $\{\mathcal{A}_\lambda\} \xrightarrow{\{f_\lambda\}} \{\mathcal{B}_\lambda\}$ be a morphism of cofiltered systems of idempotent complete additive ∞-categories which is levelwise additive and essentially surjective. Assume $\{\mathcal{A}_\lambda\}$ has enough objects given by a set* S. *If the map*

$$\text{“}\lim\text{”}\, \mathrm{Maps}_{\mathcal{A}_\lambda}(s_\lambda, s'_\lambda) \to \text{“}\lim\text{”}\, \mathrm{Maps}_{\mathcal{B}_\lambda}(f(s_\lambda), f(s'_\lambda))$$

*is a weak equivalence of pro-$\mathrm{S}\times\mathrm{S}$-indexed spaces, then “*lim*”f_λ is also a weak equivalence.*

Proof. It follows from Theorem 6.3.7 that the maps

$$\{\tau_{\leqslant n}\mathcal{A}_\lambda\} \xrightarrow{\{f_\lambda\}} \{\tau_{\leqslant n}\mathcal{B}_\lambda\}$$

are pro-equivalences. Hence we obtain the result. □

We can now strengthen Theorem 6.3.7 a little bit.

Corollary 6.3.10. *Let $\{\mathcal{A}_\lambda\} \xrightarrow{\{f_\lambda\}} \{\mathcal{B}_\lambda\}$ be a morphism of* pro-truncated *(see Definition 6.3.3) cofiltered systems of idempotent complete additive ∞-categories which is levelwise additive and essentially surjective. Assume $\{\mathcal{A}_\lambda\}$ has enough objects given by a set* S. *If the map*

$$\text{“}\lim\text{”}\, \mathrm{Maps}_{\mathcal{A}_\lambda}(s_\lambda, s'_\lambda) \to \text{“}\lim\text{”}\, \mathrm{Maps}_{\mathcal{B}_\lambda}(f(s_\lambda), f(s'_\lambda))$$

*is an equivalence of pro-$\mathrm{S}\times\mathrm{S}$-indexed spaces, then “*lim*”f_λ is also an equivalence.*

Proof. For any locally bounded $\{\mathcal{A}_\lambda\}$ the object “lim”$\mathcal{A}_\lambda$ is a limit of (constant) bounded objects. Hence the maps

$$\text{“}\lim\text{”}\mathcal{A}_\lambda \to \tau_{<\infty}\text{“}\lim\text{”}\mathcal{A}_\lambda$$

$$\text{“}\lim\text{”}\mathcal{B}_\lambda \to \tau_{<\infty}\text{“}\lim\text{”}\mathcal{B}_\lambda$$

are pro-equivalences. The map

$$\tau_{<\infty}\text{“}\lim\text{”}\mathcal{A}_\lambda \to \tau_{<\infty}\text{“}\lim\text{”}\mathcal{B}_\lambda$$

is equivalence by Corollary 6.3.9 and [KST18, Lemma 2.7], so we obtain the result. □

Finally by combining Corollary 6.3.10 and Theorem 3.2.7 we obtain the following criterion for pro-equivalence of weighted stable ∞-categories.

Corollary 6.3.11. *Any morphism* $\{\mathcal{A}_\lambda\} \overset{\{f_\lambda\}}{\to} \{\mathcal{B}_\lambda\}$ *of cofiltered systems of idempotent complete boundedly weighted stable* ∞*-categories whose associated cofiltered systems of hearts are pro-truncated and have enough objects given by a set* S *is a pro-equivalence whenever the map*

$$\text{``}\lim\text{''}\,\mathrm{Maps}_{\mathcal{A}_\lambda}(s_\lambda, s'_\lambda) \to \text{``}\lim\text{''}\,\mathrm{Maps}_{\mathcal{B}_\lambda}(f_\lambda(s_\lambda), f_\lambda(s'_\lambda))$$

is an equivalence in $\mathrm{Pro}(\mathrm{Spc}^{\mathrm{S}\times\mathrm{S}})$.

Corollary 6.3.12. *Any morphism* $\{\mathcal{A}_\lambda\} \overset{\{f_\lambda\}}{\to} \{\mathcal{B}_\lambda\}$ *of cofiltered systems of idempotent complete boundedly weighted stable* ∞*-categories whose associated cofiltered systems of hearts have enough objects given by a set* S *is a weak equivalence whenever the map*

$$\text{``}\lim\text{''}\,\mathrm{Maps}_{\mathcal{A}_\lambda}(s_\lambda, s'_\lambda) \to \text{``}\lim\text{''}\,\mathrm{Maps}_{\mathcal{B}_\lambda}(f_\lambda(s_\lambda), f_\lambda(s'_\lambda))$$

is a weak equivalence in $\mathrm{Pro}(\mathrm{Spc}^{\mathrm{S}\times\mathrm{S}})$.

Remark 6.3.13. We do not know how to detect pro-equivalence of categories or ∞-categories which are not additive. Moreover, we do not know whether there is an analogue of Corollary 6.3.7 for cofiltered systems of stable ∞-categories that have enough objects in an appropriate sense (i.e. all categories should be generated under finite limits, finite colimits and retracts by a chosen set of objects S).

Corollary 6.3.7 is crucial for the proof of pro-excision on stacks and we would be able to prove it in a much greater generality if there was an analogue for arbitrary stable ∞-categories.

6.4. **Pro-Milnor squares.** This goal of this part is to make sense of [LT19, Section 2.4] in the context of stable ∞-categories. In general it is hard to deal with equivalences in $\mathrm{Pro}(\mathbf{Cat}^{\mathrm{st}}_\infty)$. However, thanks to the results of the previous section we understand equivalences in $\mathrm{Pro}(\mathbf{WCat}^{\mathrm{st},b}_\infty)$ quite well. Hence in this subsection we restrict our attention to boundedly weighted stable ∞-categories. This is completely analogous to the connectivity assumption in [LT19, Section 2.4]. Consider a commutative square of cofiltered diagrams of boundedly weighted stable ∞-categories

$$\begin{array}{ccc} \{\mathcal{A}_\lambda\} & \xrightarrow{\{f_\lambda\}} & \{\mathcal{B}_\lambda\} \\ \big\downarrow{\scriptstyle\{p_\lambda\}} & & \big\downarrow{\scriptstyle\{q_\lambda\}} \\ \{\mathcal{A}'_\lambda\} & \xrightarrow{\{g_\lambda\}} & \{\mathcal{B}'_\lambda\}. \end{array} \tag{6.4.1}$$

Assume that the functors $f_\lambda, p_\lambda, q_\lambda, g_\lambda$ are dense for all λ and that $\{\mathcal{A}^\heartsuit_\lambda\}$ has enough objects given by a set S.

Definition 6.4.2. We say that it is a **pro-Milnor square** if the following properties hold:

(1) the stable ∞-category $\mathcal{A}'_\lambda \times^{\mathcal{A}_\lambda}_{\mathcal{B}'_\lambda} \mathcal{B}_\lambda$ (see Definition 6.1.12) is boundedly weighted in a way that makes the functor $\mathcal{A} \to \mathcal{A}'_\lambda \times^{\mathcal{A}_\lambda}_{\mathcal{B}'_\lambda} \mathcal{B}_\lambda$ weight-exact,

(2) the map $\{\mathcal{A}_\lambda\} \to \{\mathcal{A}'_\lambda \times^{\mathcal{A}_\lambda}_{\mathcal{B}'_\lambda} \mathcal{B}_\lambda\}$ is a pro-equivalence,

(3) the map $\{\mathcal{A}'_\lambda \odot^{\mathcal{B}'_\lambda}_{\mathcal{A}_\lambda} \mathcal{B}_\lambda\} \to \{\mathcal{B}'_\lambda\}$ is a pro-equivalence.

We note that the weight structure in (1) is uniquely determined, as its heart must be the full subcategory of $\mathcal{A}'_\lambda \times^{\mathcal{A}_\lambda}_{\mathcal{B}'_\lambda} \mathcal{B}_\lambda$ generated by the image of $\mathcal{A}^\heartsuit$ (see Remark 3.1.3).

Informally, part (2) of the definition corresponds to the condition of being a map-pullback in the definition of a Milnor square of stable ∞-categories. Part (3) corresponds to the condition of satisfying base change. Indeed, if all hearts of the weighted stable ∞-categories are n-categories then Corollary 6.3.11 allows us to rewrite these conditions more explicitly:

Proposition 6.4.3. *Assume that for any λ there exists an integer n such that $\mathcal{A}_\lambda^\heartsuit$, $\mathcal{A'}_\lambda^\heartsuit$, $\mathcal{B}_\lambda^\heartsuit$ and $\mathcal{B}_\lambda^{'\heartsuit}$ are n-categories. Then Definition 6.4.2 is equivalent to the following*

(1) The spectra

$$\mathbf{Maps}_{\mathcal{B}_\lambda}(f_\lambda(s_\lambda), f_\lambda(s'_\lambda)) \times_{\mathbf{Maps}_{\mathcal{B}'_\lambda}(q_\lambda f_\lambda(s_\lambda), q_\lambda f_\lambda(s'_\lambda))} \mathbf{Maps}_{\mathcal{A}'_\lambda}(p_\lambda(s_\lambda), p_\lambda(s'_\lambda))$$

are connective for all λ, $s \in \mathrm{S}$,

(2) The map

$$\begin{array}{c} \text{“}\lim\text{”}\, \mathrm{Maps}_{\mathcal{A}_\lambda}(s_\lambda, s'_\lambda) \\ \big\downarrow \\ \text{“}\lim\text{”}\, \mathrm{Maps}_{\mathcal{B}_\lambda}(f_\lambda(s_\lambda), f_\lambda(s'_\lambda)) \times_{\mathrm{Maps}_{\mathcal{B}'_\lambda}(q_\lambda f_\lambda(s_\lambda), q_\lambda f_\lambda(s'_\lambda))} \mathrm{Maps}_{\mathcal{A}'_\lambda}(p_\lambda(s_\lambda), p_\lambda(s'_\lambda)) \end{array}$$

is an equivalence in $\mathrm{Pro}(\mathrm{Spc}^{\mathrm{S}\times\mathrm{S}})$*,*

(3) the map

$$\text{“}\lim\text{”}\, \mathrm{Maps}_{\mathcal{A}'_\lambda \odot^{\mathcal{B}'_\lambda}_{\mathcal{A}_\lambda} \mathcal{B}_\lambda}(q'_\lambda f_\lambda(s_\lambda), q'_\lambda p_\lambda f_\lambda(s'_\lambda)) \longrightarrow \text{“}\lim\text{”}\, \mathrm{Maps}_{\mathcal{B}'_\lambda}(q_\lambda f_\lambda(s_\lambda), q_\lambda p_\lambda f_\lambda(s'_\lambda))$$

is an equivalence in $\mathrm{Pro}(\mathrm{Spc}^{\mathrm{S}\times\mathrm{S}})$*, where q'_λ denotes the natural map $\mathcal{B}'_\lambda \to \mathcal{A}'_\lambda \odot^{\mathcal{B}'_\lambda}_{\mathcal{A}_\lambda} \mathcal{B}_\lambda$.*

Proof. It follows from Corollary 6.3.11 that the conditions in Definition 6.4.2 follow from the corresponding conditions in the theorem. In fact, these conditions also follow from Definition 6.4.2 since the functors

$$\mathrm{Pro}(\mathbf{WCat}_\infty^{\mathrm{st},b})_{\mathrm{Fun}(\mathrm{S},\mathrm{Spt}^{\mathrm{fin}})/} \to \mathrm{Pro}(\mathbf{WCat}_\infty^{\mathrm{st},b})$$

are conservative[10] and the maps in $\mathrm{Pro}(\mathrm{Spc}^{\mathrm{S}\times\mathrm{S}})$ are functorial images of the corresponding maps in $\mathrm{Pro}(\mathbf{WCat}_\infty^{\mathrm{st},b})_{\mathrm{Fun}(\mathrm{S},\mathrm{Spt}^{\mathrm{fin}})/}$. ☐

6.4.4. *Pro-excision for stable ∞-categories.* Any localizing invariant E induces a functor

$$\mathrm{Pro}(\mathbf{WCat}_\infty^{\mathrm{st},b}) \to \mathrm{Pro}(\mathrm{Spt}).$$

Our main result is the following:

Theorem 6.4.5. *Assume given a pro-Milnor square (6.4.1) such that for any λ there exists an integer n such that $\mathcal{A}_\lambda^\heartsuit$, $\mathcal{A'}_\lambda^\heartsuit$, $\mathcal{B}_\lambda^\heartsuit$ and $\mathcal{B}_\lambda^{'\heartsuit}$ are n-categories. Then any localizing invariant sends it to a pullback square in pro-spectra.*

Proof. Consider the diagram

$$\begin{array}{ccccc} \{\mathcal{A}_\lambda\} & \longrightarrow & \{\mathcal{A}'_\lambda \times^{\mathcal{A}_\lambda}_{(\mathcal{A}'_\lambda \odot^{\mathcal{B}'_\lambda}_{\mathcal{A}_\lambda} \mathcal{B}_\lambda)} \mathcal{B}_\lambda\} & \longrightarrow & \{\mathcal{B}_\lambda\} \\ & & \big\downarrow & & \big\downarrow \\ & & \{\mathcal{A}'_\lambda\} & \longrightarrow & \{\mathcal{A}'_\lambda \odot^{\mathcal{B}'_\lambda}_{\mathcal{A}_\lambda} \mathcal{B}_\lambda\} & \longrightarrow & \{\mathcal{B}'_\lambda\}. \end{array}$$

The square in the middle is levelwisely a weak pullback square and all the functors at all levels are dense by construction. Moreover, Construction 6.1.8 only depends on the image of the left upper corner, so the base change transformation in the square is an equivalence (see Proposition 6.2.3). Hence the square is a levelwise Milnor square and by Theorem 6.2.4 any

[10]the functor $\mathcal{C}_{\mathrm{X}/} \to \mathcal{C}$ is conservative for any ∞-category $\mathcal{C}$ and its object X (this follows from the classical statement and [Lur17b, Remark 1.2.9.6])

localizing invariant sends it to a levelwise pullback square. By Lemma 6.3.1 this gives rise to a pullback in Pro(Spt).

The rightmost horizontal morphism is a pro-equivalence by assumption. So it suffices to show that

$$\{\mathcal{A}_\lambda\} \to \{\mathcal{A}'_\lambda \times^{\mathcal{A}_\lambda}_{(\mathcal{A}'_\lambda \odot^{\mathcal{B}'_\lambda}_{\mathcal{A}_\lambda} \mathcal{B}_\lambda)} \mathcal{B}_\lambda\}$$

is a pro-equivalence. By Corollary 6.3.11 it suffices to show that the map (1) in the diagram

$$\begin{array}{c} \text{“}\lim\text{''}\, \mathrm{Maps}_{\mathcal{A}_\lambda}(s_\lambda, s'_\lambda) \\ \downarrow{\scriptstyle (1)} \\ \text{“}\lim\text{''}\, \mathrm{Maps}_{\mathcal{B}_\lambda}(f_\lambda(s_\lambda), f_\lambda(s'_\lambda)) \times_{\mathrm{Maps}_{(\mathcal{A}'_\lambda \odot^{\mathcal{B}'_\lambda}_{\mathcal{A}_\lambda} \mathcal{B}_\lambda)}(q'_\lambda f_\lambda(s_\lambda), q'_\lambda f_\lambda(s'_\lambda))} \mathrm{Maps}_{\mathcal{A}'_\lambda}(p_\lambda(s_\lambda), p_\lambda(s'_\lambda)) \\ \downarrow{\scriptstyle (2)} \\ \text{“}\lim\text{''}\, \mathrm{Maps}_{\mathcal{B}_\lambda}(f_\lambda(s_\lambda), f_\lambda(s'_\lambda)) \times_{\mathrm{Maps}_{\mathcal{B}'_\lambda}(q_\lambda f_\lambda(s_\lambda), q_\lambda f_\lambda(s'_\lambda))} \mathrm{Maps}_{\mathcal{A}'_\lambda}(p_\lambda(s_\lambda), p_\lambda(s'_\lambda)) \end{array}$$

is an equivalence. The composite is an equivalence by the first part of Proposition 6.4.3(1) and the map (2) is an equivalence by the second part of Proposition 6.4.3(2) combined with Lemma 6.3.1. □

6.4.6. *Pro-base change map.* In this subsection we prove that part (3) in Definition 6.4.2 can be reformulated in terms of base change maps.

Proposition 6.4.7. *Assume that for any λ there exists an integer n such that $\mathcal{A}^\heartsuit_\lambda$, $\mathcal{A}'^\heartsuit_\lambda$, $\mathcal{B}^\heartsuit_\lambda$ and $\mathcal{B}'^\heartsuit_\lambda$ are n-categories. Assume also that parts (1) and (2) of Definition 6.4.2 are satisfied. Then part (3) is equivalent to saying that the map in* $\mathrm{Pro}(\mathrm{Spc}^{\mathrm{S}\times\mathrm{S}})$*:*

$$\text{“}\lim\text{''}\, \mathrm{Maps}_{\mathrm{Ind}(\mathcal{B}_\lambda)}(f^*_\lambda(s_\lambda), f^*_\lambda p_{\lambda,*} p^*_\lambda(s'_\lambda)) \longrightarrow \text{“}\lim\text{''}\, \mathrm{Maps}_{\mathrm{Ind}(\mathcal{B}_\lambda)}(f^*_\lambda(s_\lambda), q_{\lambda,*} g^*_\lambda p^*_\lambda(s'_\lambda))$$

is an equivalence.

Before proving this technical lemma we will need to introduce a notion of a morphism in cofiltered systems of ∞-categories.

Definition 6.4.8. A **uniform morphism** in $\{\mathcal{A}_\lambda\}$ is a natural transformation between functors from $\{\mathrm{S}\}$ to $\{\mathcal{A}_\lambda\}$, where S is a set. We say that a uniform morphism $\{f_\lambda\} : \{s_\lambda\} \to \{t_\lambda\}$ **admits a pro-inverse** if there exists an increasing function (i.e. a function on objects with a morphism $i \to \varphi(i)$)

$$\varphi : \mathrm{I} \to \mathrm{I}$$

and $h \in \lim_\lambda \mathrm{Maps}_{\mathcal{A}_\lambda}(t_\lambda, s_{\varphi(\lambda)})$ with homotopies

$$h \circ \lim f_\lambda \simeq \mathrm{id}_{s_\lambda} \text{ in } \lim_\lambda \mathrm{Maps}_{\mathcal{A}_\lambda}(s_\lambda, s_{\varphi(\lambda)})$$

$$\lim f_\lambda \circ h \simeq \mathrm{id}_{t_\lambda} \text{ in } \lim_\lambda \mathrm{Maps}_{\mathcal{A}_\lambda}(t_\lambda, t_{\varphi(\lambda)})$$

where the limit on the right is taken in the ∞-category $\mathrm{Spc}^{\mathrm{S}\times\mathrm{S}}$.

Lemma 6.4.9. *Let $\{\mathcal{A}_\lambda\}$ be a cofiltered system of idempotent complete additive ∞-categories that has enough objects given by a set* S*. Let $\{x_{s,\lambda}\} \overset{\{f_\lambda\}}{\to} \{y_{s,\lambda}\}$ be an* S*-uniform morphism in $\{\mathcal{A}_\lambda\}$ that admits a pro-inverse. Then the map*

$$\text{“}\lim\text{''}\, \mathrm{Maps}_{\mathcal{A}_\lambda}(s_\lambda, x_{s',\lambda}) \to \text{“}\lim\text{''}\, \mathrm{Maps}_{\mathcal{A}_\lambda}(s_\lambda, y_{s',\lambda})$$

is an equivalence in $\mathrm{Pro}(\mathrm{Spc}^{\mathrm{S}\times\mathrm{S}})$.

Proof. It suffices to observe that the spaces

$$\lim_{\lambda} \mathrm{Maps}_{\mathcal{A}_\lambda}(x_{s,\lambda}, y_{s,\lambda})$$

$$\lim_{\lambda} \mathrm{Maps}_{\mathcal{A}_\lambda}(x_{s,\lambda}, x_{\varphi(s),\lambda})$$

$$\lim_{\lambda} \mathrm{Maps}_{\mathcal{A}_\lambda}(y_{s,\lambda}, x_{\varphi(s),\lambda},)$$

$$\lim_{\lambda} \mathrm{Maps}_{\mathcal{A}_\lambda}(y_{s,\lambda}, y_{\varphi(s),\lambda})$$

act on the corresponding objects of $\mathrm{Pro}(\mathrm{Spc}^{\mathrm{S}\times\mathrm{S}})$ in a way compatible with the composite and the identity. $\square$

Proof of Proposition 6.4.7. We first note that the construction

$$\mathcal{A}' \odot^{\mathcal{B}'}_{\mathcal{A}} \mathcal{B}$$

only depends on the image of $\mathcal{A}_\lambda$ in the lax pullback, in particular it can be identified with

$$\mathcal{A}' \odot^{\mathcal{B}'}_{\left(\mathcal{A}' \times^{\mathcal{A}}_{(\mathcal{A}'_\lambda \odot^{\mathcal{B}'}_{\mathcal{A}} \mathcal{B})} \mathcal{B}\right)} \mathcal{B}.$$

It follows from Proposition 6.4.3 and Theorem 6.1.15 that part (3) in Definition 6.4.2 is equivalent to the map in $\mathrm{Pro}(\mathrm{Spc}^{\mathrm{S}\times\mathrm{S}})$

$$\begin{array}{c} \text{“}\lim\text{”} \mathrm{Maps}_{\mathrm{Ind}(\mathcal{B}_\lambda)}(f^*_\lambda(s_\lambda), f'^*_\lambda p'_{\lambda,*} p^*_\lambda(s'_\lambda)) \\ \big\downarrow \\ \text{“}\lim\text{”} \mathrm{Maps}_{\mathrm{Ind}(\mathcal{B}_\lambda)}(f^*_\lambda(s_\lambda), q_{\lambda,*} g^*_\lambda p^*_\lambda(s'_\lambda)) \end{array}$$

being an equivalence, where p'_λ and f'_λ denote the functors

$$\mathcal{A}'_\lambda \times^{\mathcal{A}_\lambda}_{\mathcal{B}'_\lambda} \mathcal{B}_\lambda \to \mathcal{A}'_\lambda,$$

$$\mathcal{A}'_\lambda \times^{\mathcal{A}_\lambda}_{\mathcal{B}'_\lambda} \mathcal{B}_\lambda \to \mathcal{B}_\lambda.$$

Denote the functor

$$\mathcal{A}_\lambda \to \mathcal{A}'_\lambda \times^{w,\mathcal{A}_\lambda}_{\mathcal{B}'_\lambda} \mathcal{B}_\lambda,$$

by h_λ. Since we have an equivalence

$$f^*_\lambda p_{\lambda,*} \simeq f'^*_\lambda h^*_\lambda h_{\lambda *} p'_{\lambda,*}$$

to prove our claim it suffices to show that the map

$$\text{“}\lim\text{”} \mathrm{Maps}_{\mathrm{Ind}(\mathcal{B}_\lambda)}(f^*_\lambda(s_\lambda), f'^*_\lambda h^*_\lambda h_{\lambda *} p'_{\lambda,*} p^*_\lambda(s'_\lambda)) \longrightarrow \text{“}\lim\text{”} \mathrm{Maps}_{\mathrm{Ind}(\mathcal{B}_\lambda)}(f^*_\lambda(s_\lambda), f'^*_\lambda p'_{\lambda,*} p^*_\lambda(s'_\lambda))$$

is an equivalence. The first condition in the Definition implies that the S-uniform morphism $\{h^*_\lambda h_{\lambda *} \mathrm{X}_\lambda\} \to \{\mathrm{X}_\lambda\}$ admits a pro-inverse for any object X of $\{\mathcal{A}'_\lambda \times^{\mathcal{A}_\lambda}_{\mathcal{B}'_\lambda} \mathcal{B}_\lambda\}$ (see Proposition 6.4.3). Hence $\{f'^*_\lambda h^*_\lambda h_{\lambda *} \mathrm{X}_\lambda\} \to \{f'^*_\lambda \mathrm{X}_\lambda\}$ also admits a pro-inverse and thus induces an equivalence on $\mathrm{S} \times \mathrm{S}$-indexed mapping spaces by Lemma 6.4.9. $\square$

6.4.10. *Weak equivalences of pro-objects.* The notion of pro-equivalence is sometimes too strong to conveniently talk about. In particular, the pro-homotopy groups (whenever those are defined) do not reflect pro-equivalences for pro-objects that are not pro-truncated (see Definition 6.3.3). Because of this, weak equivalences in the sense of Definition 6.3.2 or similar notions are used instead in the literature (see [KST17, Definition 4.4] and [LT19, Definition 2.7]). In this section we use them to prove an unbounded version of Theorem 6.4.5. The category of perfect complexes over a $\mathbb{E}_1$-ring spectrum admits a weight structure, so our theory generalizes the results of [LT19, 2.4] automatically.

Definition 6.4.11. A localizing invariant E is said to be **connected** if for any map of boundedly weighted stable ∞-categories $\mathcal{A} \to \mathcal{B}$ such that $\tau_{\leqslant n}\mathcal{A} \to \tau_{\leqslant n}\mathcal{B}$ is an equivalence, the map of spectra $\mathrm{E}(\mathcal{A}) \to \mathrm{E}(\mathcal{B})$ is n-connected.

Remark 6.4.12. The notion of a connected invariant is closely related to the notion of an *n-connective invariant* in [LT19, Def. 2.5]. In fact, any functor $\mathcal{A} \xrightarrow{\mathrm{F}} \mathcal{B}$ with $\tau_{\leqslant n}\mathcal{A} \to \tau_{\leqslant n}\mathcal{B}$ being an equivalence, can be presented as a filtered colimit of functors

$$\mathbf{Perf}_{\mathbf{End}_{\mathcal{A}}(\bigoplus_{i=1}^{n} \mathrm{X}_i)} \to \mathbf{Perf}_{\mathbf{End}_{\mathcal{A}}(\mathrm{F}(\bigoplus_{i=1}^{n} \mathrm{X}_i))}$$

induced by n-connected maps of rings. So for a finitary invariant such as K-theory, the property of being connected is equivalent to being 0-connective. In general, an argument similar to the one used for proving Theorem 5.3.3 implies that any localizing invariant E is connected whenever it is 1-connective.

Remark 6.4.13. For a connected invariant E the associated functor

$$\mathrm{Pro}(\mathbf{WCat}_{\infty}^{\mathrm{st},b}) \to \mathrm{Pro}(\mathrm{Spt})$$

sends weak equivalences to weak equivalences.

Theorem 6.4.14. *Assume given a square (6.4.1) such that for any λ there exists an integer n such that $\mathcal{A}_{\lambda}^{\heartsuit}$, $\mathcal{A}'^{\heartsuit}_{\lambda}$, $\mathcal{B}_{\lambda}^{\heartsuit}$ and $\mathcal{B}'^{\heartsuit}_{\lambda}$ are n-categories. Given a pro-Milnor square (6.4.1), any localizing invariant sends it to a pullback square in pro-spectra.*

Proof. The proof of Theorem 6.4.5 works verbatim after replacing pro-equivalences with weak equivalences and using Corollary 6.3.12 instead of Corollary 6.3.11. ☐

7. Applications to localizing invariants of ANS stacks

We are now going to apply all the abstract results of the previous section to stacks and their algebraic invariants. Our first goal is to prove the pro-cdh-descent for K-theory of algebraic stacks (Theorem 7.3.1). The proof will follow the strategy of [KST17] with the main new ingredient being the equivariant pro-descent. This allows us to prove the Weibel's conjecture on the vanishing of the negative K-theory of stacks. Besides that, Theorem 7.3.1 together with Theorem 1.4.7 implies cdh-descent for all truncating invariants on stacks. Finally, we will present an application of Theorem 1.4.7 to the lattice conjecture.

From now on we will restrict our attention to *derived* quasi-compact and quasi-separated algebraic stacks. The reason for this choice is the fact that the formal completion of a closed subscheme is bounded in this setting. This allows us to use the categorical pro-descent results of Section 6.4. We note that the generalization described in Section 6.4.10 allows us to obtain a statement for connected invariants of spectral stacks.

7.1. **ANS stacks.** Here we introduce a large class of stacks to which our results apply. All applications of our categorical results to localizing invariants are obtained by first doing the case of affine quotient stacks (i.e. stacks described via Construction 3.3.12) and then using the so-called Nisnevich topology to glue a result for more general stacks. So, the stacks we care about are Nisnevich-locally affine quotient stacks. Informally, the class of ANS stacks is the largest class of such stacks that is closed under reasonable operations (such as blow-ups).

7.1.1. *Nisnevich topology.*

Definition 7.1.2. Let X be a derived stack. We say that a family of etale maps $\{\mathrm{U}_i \to \mathrm{X}\}_{i\in \mathrm{I}}$ is a **Nisnevich covering** if for every point $x \in \mathrm{X}$ there exists $i \in \mathrm{I}$ and a point $y \in \mathrm{U}_i$ over x with an isomorphism of residual gerbes. This generates a topology called the **Nisnevich topology** on the small etale site of X.

A **Nisnevich square** of derived algebraic stacks is a cartesian diagram of derived algebraic stacks

$$\begin{array}{ccc} \mathrm{U}' & \longrightarrow & \mathrm{X}' \\ \downarrow & & \downarrow{\scriptstyle f} \\ \mathrm{U} & \xrightarrow{j} & \mathrm{X}. \end{array}$$

where j is a quasi-compact, quasi-separated open immersion and f is a representable étale morphism of finite presentation inducing an equivalence $f^{-1}(\mathrm{Z}) \to \mathrm{Z}$ for some complementary (to U) closed substack $\mathrm{Z} \hookrightarrow \mathrm{X}$.

Recall some basic properties of these notions:

(1) Any Nisnevich covering has a finite subcovering (the proof is the same as in the discrete case [HK19, Corollary 2.7]).
(2) Nisnevich squares define a cd-structure on the category of stacks. The topology generated by this cd-structure coincides with the Nisnevich topology (the proof is the same as in the discrete case [HK19, Proposition 2.9]). Moreover, a presheaf F satisfies descent in this topology if and only if F maps *all* Nisnevich squares into pullback squares ([AHW17, Theorem 3.2.5]).

Definition 7.1.3. A derived algebraic stack X is **perfect** if the stable ∞-category $\mathbf{D}_{\mathrm{qc,X}}$ is compactly generated by its full subcategory $\mathbf{Perf}_{\mathrm{X}}$ of perfect complexes.

Example 7.1.4. We have already seen that the stacks of the form $[\mathrm{Spec}\,\mathrm{R}/\mathrm{G}]$ are examples of perfect stacks when G is linearly reductive or affine over a field of characteristic 0. (see Proposition 3.3.15 and Theorem 3.3.13).

7.1.5. *Nisnevich descent for localizing invariants.* Similarly to the case of schemes, reasonable cohomology theories tend to satisfy descent in the Nisnevich topology. Let E be a localizing invariant. For a stack X adopt the notation

$$\mathrm{E}(\mathrm{X}) := \mathrm{E}(\mathbf{Perf}_{\mathrm{X}}).$$

Thus a localizing invariant gives rise to a presheaf of spectra on stacks.

Theorem 7.1.6. *Any localizing invariant* E *sends a Nisnevich square*

$$\begin{array}{ccc} \mathrm{U} \times_{\mathrm{X}} \mathrm{V} & \longrightarrow & \mathrm{V} \\ \downarrow & & \downarrow{\scriptstyle p} \\ \mathrm{U} & \xrightarrow{i} & \mathrm{X}. \end{array}$$

with perfect X *and* V *to a cartesian square of spaces.*

Proof. The induced square

$$\begin{array}{ccc} \mathbf{D}_{\mathrm{qc,X}} & \xrightarrow{j} & \mathbf{D}_{\mathrm{qc,U}} \\ \downarrow & & \downarrow \\ \mathbf{D}_{\mathrm{qc,V}} & \longrightarrow & \mathbf{D}_{\mathrm{qc,U\times_X V}} \end{array}$$

is cartesian. This is [TT90, Thm 2.6.3] in the classical setting. In fact, one can check the invertibility of the unit and co-unit of the adjunction

$$\mathbf{D}_{\mathrm{qc,X}} \rightleftarrows \mathbf{D}_{\mathrm{qc,U}} \times_{\mathbf{D}_{\mathrm{qc,U\times_X V}}} \mathbf{D}_{\mathrm{qc,V}}$$

using base change for i_* and p_* ([Lur18, Cor 3.4.2.2]) and the semi-orthogonal decomposition

$$\mathbf{D}_{\mathrm{qc,X}} = \langle \mathbf{D}_{\mathrm{qc,X}^\wedge_{(\mathrm{X-U})}}, \mathbf{D}_{\mathrm{qc,U}} \rangle,$$

see [GR14, 7.1] or [HLP14, Thm 2.2.3]. The functors j_* and i_* right adjoint to the horizontal functors in the diagram are fully faithful by base change ([Lur18, Cor 3.4.2.2]). Now the result follows from [Tam18, Thm 18] (see also [Hoy18, Cor 13]) and the perfectness assumption. □

Definition 7.1.7. A derived algebraic stack X is called **ANS** [11] if it has affine diagonal and nice stabilizers.

Example 7.1.8. Let G be a finite étale group scheme over a field k. If G has order prime to the characteristic of k, then G is nice and embeddable. It follows that any separated Deligne–Mumford stack over k is ANS as long as it is tame (i.e., has all stabilizers of order prime to the characteristic).

Example 7.1.9. Any algebraic stack with affine diagonal that is tame in the sense of [AOV08, Def. 3.1] is ANS. This generalizes Example 7.1.8.

Example 7.1.10. If T is a nice group scheme over an affine scheme S acting on an algebraic space X over S with affine diagonal, then the quotient [X/T] is ANS. (However, it is typically not tame in the sense of [AOV08].)

Lemma 7.1.11. *Let* X *be an ANS derived stack. Let* $f : \mathrm{X}' \to \mathrm{X}$ *be a representable morphism with affine diagonal. Then* X′ *is ANS.*

Proof. Since f is representable, the stabilizers of X′ are subgroups of those of X. □

The following is the main result of [AHHLR] in the classical case. The generalization to derived stacks is immediate.

Theorem 7.1.12 (Alper–Hall–Halpern-Leistner–Rydh)**.** *Let* X *be an ANS derived stack. Then there exists a finite sequence of open immersions*

$$\varnothing = \mathrm{U}_0 \hookrightarrow \mathrm{U}_1 \hookrightarrow \cdots \hookrightarrow \mathrm{U}_n = \mathrm{X}, \tag{7.1.13}$$

an embeddable nice group scheme G *over an affine scheme* S*, and Nisnevich squares*

$$\begin{array}{ccc} \mathrm{W}_i & \longrightarrow & \mathrm{V}_i \\ \downarrow & & \downarrow \\ \mathrm{U}_{i-1} & \longrightarrow & \mathrm{U}_i \end{array}$$

where V_i *is étale and affine over* U_i *and affine over* BG*.*

[11] This is an abbreviation of "affine diagonal and nice stabilizers."

Proof. If X is classical, this follows by combining the main result of [AHHLR][12] with [HK19, Prop. 2.9]. In general, we can use derived invariance of the étale site [Lur17a, Thm. 7.5.0.6] to obtain unique lifts of the U_i and V_i from the classical truncation X_{cl}, in such a way that we have Nisnevich squares as above. It remains only to extend the affine morphism $(V_i)_{cl} \to BG$ to V_i. In fact, the relevant obstruction vanishes since the cotangent complex of BG is of Tor-amplitude $[0, 1]$ and $(V_i)_{cl}$ is cohomologically affine (as G is linearly reductive). □

Theorem 7.1.14. *Let* X *be an ANS derived stack. Then* X *is perfect.*

Proof. By Theorem 7.1.12, we can in particular find an affine étale surjection onto X from a finite coproduct of quotient stacks of the form $[X_i/G]$, where G is a nice group scheme over an affine scheme S and X_i is an affine derived S-scheme with G-action. It will now suffice to show that the property of crispness can be detected by affine étale surjections. Indeed, we note that the proof given in [HR15b, Thm. C] for the classical case generalizes to our setting, following Example 9.4 of *loc. cit.* □

Combining Theorem 7.1.14, Theorem 7.1.6 and Lemma 7.1.11 we obtain:

Corollary 7.1.15. *Any localizing invariant* E *satisfies Nisnevich descent on any ANS derived stack.*

7.2. **Dimension of algebraic stacks.** Recall several useful notions of dimension from [HK19].

Definition 7.2.1. Let X be a Noetherian stack.

(1) The *Krull dimension* $\dim X$ of X is the dimension of the underlying topological space $|X|$.
(2) We define the *blow-up dimension* $\operatorname{bl}\dim X$ to be the supremum of such n that there exists a sequence of maps
$$X_n \to \cdots \to X_0 = X$$
such that X_i is a nonempty nowhere dense closed substack in an iteratated blow-up of X_{i-1}. If X is empty, then $\operatorname{bl}\dim(X) = -1$ by convention.
(3) The *covering dimension* $\operatorname{cov}\dim_{fppf} X$ is the minimum of all such n that there exists a faithfully flat finitely presented morphism $S \to X$ where S a scheme of Krull dimension n.
(4) The *covering dimension* $\operatorname{cov}\dim_{sm}(X)$ (more precisely, *smooth-covering dimension*) is the minimal integer $-1 \leqslant n \leqslant \infty$ such that there exists a smooth surjection $X \to X$ where X is a noetherian scheme of Krull dimension n.

Remark 7.2.2. In general, one has the inequalities $\dim(X) \leqslant \operatorname{bl}\dim(X) \leqslant \operatorname{cov}\dim_{fppf}(X) \leqslant \operatorname{cov}\dim_{sm}(X)$. For quasi-Deligne–Mumford stacks these are all equal to the usual dimension as defined in [The17, Tag 0AFL]. See [HK19, Lemma 7.8].

7.2.3. *Nisnevich cohomological dimension.* We denote by $\operatorname{cd}_{Nis} X$ the Nisnevich cohomological dimension of a stack X.

We are now going to show that the Nisnevich cohomological dimension of an algebraic stack is bounded by its covering dimension. We establish this by showing that the Nisnevich topology as a *cd*-topology on the category of Noetherian algebraic stacks of finite covering dimension is bounded with respect to a density structure defined using the density structure of Krishna and Ostvaer [KO10, Definition 8.3] on any of its faithfully flat cover. The Nisnevich cd-structure is complete and regular as Nisnevich squares are closed under pullbacks and since open

[12]See [AHR19, Cor. 17.3] and [AOV08, Thm. 3.2] for documented special cases of this result.

immersions and etale morphisms are preserved under base change and formation of diagonals. Let Stk denote the category of Noetherian algebraic stacks.

Let X be a Noetherian algebraic stack and let $\mathrm{S} \to \mathrm{X}$ be a faithfully flat finitely presented morphism from a DM-stack S. For $\mathrm{Y} \in \mathrm{Stk}/\mathrm{X}$, let $q : \mathrm{S_Y} \to \mathrm{Y}$ be the fppf covering of Y given by the base change of $\mathrm{S} \to \mathrm{X}$ along Y. For $i \geqslant 0$, let $\mathrm{D}_i^{\mathrm{S}}(\mathrm{Y})$ denote the class of open substacks $\mathrm{U} \hookrightarrow \mathrm{Y}$ such that the open substack $\mathrm{U} \times_{\mathrm{Y}} \mathrm{S_Y} \hookrightarrow \mathrm{S_Y}$ defines an element of the class $\mathrm{D}_i(\mathrm{S_Y})$, where $\mathrm{D}_*(-)$ denotes the density structure on the category of Deligne-Mumford stacks (see [KO10, Definition 8.3]). It is easy to check that the classes $\mathrm{D}_i^{\mathrm{S}}(-)$ define a density structure on the category Stk/X in the sense of [Voe10a, Definition 2.20] and that this density structure is locally of finite dimension and the dimension of X with respect to the density structure $\dim_{\mathrm{D^S}}(\mathrm{X}) = \dim(\mathrm{S})$.

Lemma 7.2.4. *Let* X *be a DM stack and* $\mathrm{U} \in \mathrm{D}_i(\mathrm{X})$. *Let* V *be an open substack of* X. *Then* $\mathrm{U} \cap \mathrm{V} \in \mathrm{D}_i(\mathrm{V})$.

Proof. By [KO10, Lemma 8.4] it suffices to show that $|\mathrm{U} \cap \mathrm{V}| \in \mathrm{D}_i(|\mathrm{V}|)$. Now we can use the same argument as in the proof of [Voe10b, Lemma 2.5]. □

Proposition 7.2.5. *The Nisnevich cd-structure on the category* Stk/X *is bounded with respect to the density structure* $\mathrm{D}_*^{\mathrm{S}}(-)$.

Proof. We need to show that every Nisnevich square is reducing with respect to the density structure. Consider a Nisnevich square Q in Stk/X:

(7.2.6)
$$\begin{array}{ccc} \mathrm{W} & \xhookrightarrow{j_{\mathrm{V}}} & \mathrm{V} \\ \downarrow & & \downarrow{\scriptstyle p} \\ \mathrm{U} & \xhookrightarrow{j} & \mathrm{Y}, \end{array}$$

where p is etale, j is an open immersion and the induced morphism $p^{-1}(\mathrm{Y} \setminus \mathrm{U}) \to (\mathrm{Y} \setminus \mathrm{U})$ is invertible. Choose $\mathrm{W}_0 \in \mathrm{D}_{i-1}^{\mathrm{S}}(\mathrm{W})$, $\mathrm{U}_0 \in \mathrm{D}_i^{\mathrm{S}}(\mathrm{U})$ and $\mathrm{V}_0 \in \mathrm{D}_i^{\mathrm{S}}(\mathrm{V})$. To show that the above square Q is reducing with respect to the density structure, we need to prove that there exists a Nisnevich square Q′ in Stk/X

(7.2.7)
$$\begin{array}{ccc} \mathrm{W}' & \hookrightarrow & \mathrm{V}' \\ \downarrow & & \downarrow \\ \mathrm{U}' & \hookrightarrow & \mathrm{Y}', \end{array}$$

and a morphism $\mathrm{Q}' \to \mathrm{Q}$ such that $\mathrm{W}' \to \mathrm{W}$ factors through $\mathrm{W}' \to \mathrm{W}_0$, $\mathrm{U}' \to \mathrm{U}$ through $\mathrm{U}' \to \mathrm{U}_0$, $\mathrm{V}' \to \mathrm{V}$ through $\mathrm{V}' \to \mathrm{V}_0$, and $\mathrm{Y}' \in \mathrm{D}_i^{\mathrm{S}}(\mathrm{Y})$ (see [Voe10a, Definition 2.21]). Applying Lemma 7.2.8 to the morphism $j \sqcup p$, we can find $\mathrm{Y}_0 \in \mathrm{D}_i^{\mathrm{S}}(\mathrm{Y})$ such that $j^{-1}(\mathrm{Y}_0) \subseteq \mathrm{U}_0$ and $p^{-1}(\mathrm{Y}_0) \subseteq \mathrm{V}_0$. Therefore by base changing (7.2.6) along $\mathrm{Y}_0 \hookrightarrow \mathrm{Y}$ and then replacing Y by Y_0 we are reduced to the case when $\mathrm{U} = \mathrm{U}_0$ and $\mathrm{V} = \mathrm{V}_0$ in (7.2.6). Note that $\mathrm{W}_0 \times_{\mathrm{Y}} \mathrm{Y}_0$ is in $\mathrm{D}_{i-1}^{\mathrm{S}}(\mathrm{U}_0 \times_{\mathrm{Y}} \mathrm{V}_0)$ by Lemma 7.2.4.

Let $\mathrm{Z} = \mathrm{W} \setminus \mathrm{W}_0$ and $\mathrm{C} = \mathrm{Y} \setminus \mathrm{U}$ and set $\mathrm{W}' = \mathrm{W}_0$, $\mathrm{U}' = \mathrm{U}$, $\mathrm{V}' = \mathrm{V} \setminus \mathrm{cl_V}(\mathrm{Z})$ and $\mathrm{Y}' = \mathrm{Y} \setminus (\mathrm{C} \cap \mathrm{cl_Y}(p \circ j_{\mathrm{V}}(\mathrm{Z})))$ in (7.2.7) to obtain the Nisnevich square Q′ with a natural morphism to Q given by inclusions. Now $\mathrm{Y}' \times_{\mathrm{Y}} \mathrm{S_Y} \to \mathrm{S_Y} \in \mathrm{D}_i(\mathrm{S_Y})$ by the proof of [KO10, Lemma 8.8]. Therefore $\mathrm{Q}' \to \mathrm{Q}$ satisfies all the required properties. □

Lemma 7.2.8. *Let* $f : \mathrm{W} \to \mathrm{Y}$ *be an etale and surjective morphism of Noetherian stacks. Then for any* $i \geqslant 0$ *and* $\mathrm{W}_0 \in \mathrm{D}_i^{\mathrm{S}}(\mathrm{W})$ *there exists* $\mathrm{Y}_0 \in \mathrm{D}_i^{\mathrm{S}}(\mathrm{Y})$ *such that* $f^{-1}(\mathrm{Y}_0) \subset \mathrm{W}_0$.

Proof. Let $\mathrm{S_W} = \mathrm{S} \times_{\mathrm{X}} \mathrm{W}$, $\mathrm{S_{W_0}} = \mathrm{S} \times_{\mathrm{X}} \mathrm{W}_0$ and let $f_{\mathrm{S}} : \mathrm{S_W} \to \mathrm{S_Y}$ denote the base change of $f : \mathrm{W} \to \mathrm{Y}$ along the fppf covering $q : \mathrm{S_Y} \to \mathrm{Y}$ and $\tilde{q} : \mathrm{S_W} \to \mathrm{W}$ denote the base change of q along f. Then by [KO10, Lemma 8.6] applied to f_{S} there exists $\tilde{\mathrm{Y}} \in \mathrm{D}_i(\mathrm{S_Y})$ such that $f_{\mathrm{S}}^{-1}(\tilde{\mathrm{Y}}) \subset \mathrm{S_{W_0}}$. Let $\mathrm{Y}_0 = q(\tilde{\mathrm{Y}})$, then $\mathrm{Y}_0 \in \mathrm{D}_i(\mathrm{Y})$ since q is open (see [LMB18, Proposition 8.6]), $\tilde{\mathrm{Y}} \subseteq \mathrm{Y}_0 \times_{\mathrm{Y}} \mathrm{S_Y}$ and $\tilde{\mathrm{Y}} \in \mathrm{D}_i(\mathrm{S_Y})$. Moreover $f^{-1}(\mathrm{Y}_0) = f^{-1}(q(\tilde{\mathrm{Y}})) \subseteq \tilde{q}(f_{\mathrm{S}}^{-1}(\tilde{\mathrm{Y}})) \subseteq \tilde{q}(\mathrm{S_{W_0}}) = \mathrm{W}_0$. □

Corollary 7.2.9. *Let* X *be a Noetherian stack and let* $\mathcal{F}$ *be a sheaf of abelian groups on the Nisnevich site of* X. *Then*

$$\mathrm{H}^i_{\mathrm{N}is}(\mathrm{X}, \mathcal{F}) = 0$$

for $i > \operatorname{cov\,dim_{fppf}} \mathrm{X}$. *In other words,* $\operatorname{cov\,dim_{fppf}} \mathrm{X} \geqslant \mathrm{cd}_{\mathrm{N}is}\,\mathrm{X}$.

7.3. **Localizing invariants of algebraic stacks.** In this section we formulate our main excision results and our version of the Weibel's conjecture. Our ultimate goal is to prove the pro-cdh-descent for K-theory of algebraic stacks (Threorem 7.3.1). The proof will follow the strategy of [KST17] with the main new ingredient being the equivariant pro-descent.

To the end of the section consider an abstract blow-up square of *noetherian* stacks

$$\begin{array}{ccc} \mathrm{E} & \longrightarrow & \tilde{\mathrm{X}} \\ \downarrow & & \downarrow{\scriptstyle p} \\ \mathrm{Y} & \xrightarrow{\ i\ } & \mathrm{X}, \end{array}$$

that is a commutative square with p proper representable and i closed immersion such that p induces an isomorphism $\tilde{\mathrm{X}} - \mathrm{E} \to \mathrm{X} - \mathrm{Y}$. Define Y_n to be the n-th infinitesimal thickening of Y in X and E_n to be the n-th infinitesimal thickening of E in $\tilde{\mathrm{X}}$.

The next few sections will be dedicated to proving the following result:

Theorem 7.3.1 (pro-cdh-descent)**.** *Assume* X *is an ANS stack. Then for any localizing invariant* E *the square*

$$\begin{array}{ccc} \mathrm{E}(\mathrm{X}) & \longrightarrow & \text{“}\lim\text{”}\, \mathrm{E}(\mathrm{Y}_n) \\ \downarrow & & \downarrow \\ \mathrm{E}(\tilde{\mathrm{X}}) & \longrightarrow & \text{“}\lim\text{”}\, \mathrm{E}(\mathrm{E}_n) \end{array}$$

is a pullback of pro-spectra.

For a map of stacks $\mathrm{Z} \to \mathrm{X}$ we denote

$$\mathrm{E}(\mathrm{X}, \mathrm{Z}) := \mathrm{Fib}(\mathrm{E}(\mathrm{X}) \to \mathrm{E}(\mathrm{Z})).$$

The conclusion of Theorem 7.3.1 can be reformulated into saying that the map

$$\text{“}\lim\text{”}\, \mathrm{E}(\mathrm{X}, \mathrm{Y}_n) \to \text{“}\lim\text{”}\, \mathrm{E}(\mathrm{X}, \mathrm{E}_n)$$

is an equivalence of pro-spectra. If we assume p to be finite, the theorem also holds for all ANS stacks. We postpone the proof till Section 7.5. Many results of this kind have already been proved by now. The most prominent paper is [KST17] where the theorem was proved for schemes. Moreover, it was proved in [HK19] that the square (7.3) is mapped into a pullback square of spectra by the homotopy K-theory.

Another related result is the cdh-descent for the so-called *derived blow-up square*. Recall that a quasi-smooth immersion of derived stacks $\mathrm{Y} \to \mathrm{X}$ is a map that locally fits into a pullback diagram

$$\begin{array}{ccc} Y & \longrightarrow & X \\ \downarrow & & \downarrow \\ X & \xrightarrow{0} & \mathbb{A}^n_X. \end{array}$$

In this situation we can define the derived blow-up $\mathrm{Bl}^{\mathbb{L}}_{Y/X}$ to be the pullback $\mathrm{Bl}_{X/\mathbb{A}^n} \times_{\mathbb{A}^n} X$. It turns out to be a well defined construction globally too (see [KR19, 4.1.6]).

Theorem 7.3.2 ([Kha18], Theorem A)**.** *Let* E *be any additive invariant. Then for a quasi-smooth closed immersion of derived algebraic stacks* $Y \to X$ *of virtual codimension more than 1 the diagram*

$$\begin{array}{ccc} E(X) & \longrightarrow & E(Y) \\ \downarrow & & \downarrow \\ E(\mathrm{Bl}^{\mathbb{L}}_{Y/X}) & \longrightarrow & E(\mathbb{P}(\mathcal{N}_{Y/X})) \end{array}$$

is cartesian.

Using this result together with the technique of flatification by blow-ups we can prove the following.

Theorem 7.3.3 (Weibel's conjecture)**.** *Let* X *be a noetherian ANS stack of covering dimension* d*. Then* $K_{-n}(X)$ *vanishes for all* $n > d$*.*

Note that we use the covering dimension instead of the blow-up dimension used in [HK19]. The reason for that is that we can only prove nilinvariance for the negative K-theory below the covering dimension while the homotopy K-theory is nilinvariant on the nose. Theorem 7.3.3 applies to all separated DM stacks with linearly reductive stabilizers. As an interesting special case we get:

Corollary 7.3.4. *Let* X *be a Noetherian scheme equipped with an action of a linearly reductive almost multiplicative group* G*. Then* $K^G_{-n}(X) = 0$ *vanishes for* $n > \dim X$*.*

7.4. **Equivariant pro-descent.** The set-up for the next theorem is the following data

- An embeddable linearly reductive group scheme G over a commutative ring k.
- A commutative k-algebra R with a G-action.
- A morphism $\{R_\lambda\} \to \{S_\lambda\}$ of $\mathbb{N}$-cofiltered systems of G-equivariant Noetherian simplicial commutative R-algebras.
- An equivariant cosection $M \xrightarrow{s} R$ of a G-equivariant locally-free R-module. We denote by $s(M)$ the image of the induced map on π_0 of the modules.
- A localizing invariant E.

Consider the derived quotients

$$R_\lambda(\lambda) = R_\lambda // s^{\otimes\lambda} = R_\lambda \otimes_{Sym_R(M)} R$$

$$S_\lambda(\lambda) = S_\lambda // s^{\otimes\lambda} = S_\lambda \otimes_{Sym_R(M)} R$$

where Sym_R denotes the free simplicial R-algebra on a module. Here the $Sym_R(M)$-module structures are induced by the zero map and by s. By functoriality of the above construction these admit an action of G which is compatible with the quotient maps.

Lemma 7.4.1. *Assume that* R_λ *and* S_λ *are bounded for every* λ*. If a morphism* $\{\mathrm{R}_\lambda\} \to \{\mathrm{S}_\lambda\}$ *as above is a pro-equivalence in* $\mathrm{Pro}(\mathrm{sCAlg}_\mathrm{R})$ *then the map*

$$\text{“}\lim\text{”}\, \mathrm{E}^\mathrm{G}([\mathrm{Spec}\,\mathrm{R}_\lambda/\mathrm{G}]) \to \text{“}\lim\text{”}\, \mathrm{E}^\mathrm{G}([\mathrm{Spec}\,\mathrm{S}_\lambda/\mathrm{G}])$$

is an equivalence.

Proof. If R_λ and S_λ have finitely many homotopy groups for any λ then the pro-objects “lim”R_λ and “lim”S_λ are pro-truncated (see Definition 6.3.3) and hence the map $\{\mathrm{R}_\lambda\} \to \{\mathrm{S}_\lambda\}$ is actually a pro-equivalence of $\mathcal{E}_\infty$-algebras. By [Isa, Lemma 13.2] (see also [LT19, Lemma 2.28]) the map is also a pro-equivalence of $\mathcal{E}_\infty$-algebras with G-action. Now the weighted stable ∞-category $\mathbf{Perf}_{[\mathrm{Spec}(-)/\mathrm{G}]}$ is functorial in $\mathcal{E}_\infty$-algebras with G-action, hence it induces a pro-equivalence of cofiltered systems of stable ∞-categories. Applying any localizing invariant to it yields a pro-equivalence of spectra and the claim.

□

Theorem 7.4.2. *Assume that* R_λ *and* S_λ *are bounded for every* λ*. Suppose that the map*

$$\{s(\mathrm{M})^\lambda \pi_i(\mathrm{R}_\lambda)\} \to \{s(\mathrm{M})^\lambda \pi_i(\mathrm{S}_\lambda)\}$$

is a pro-equivalence for all $i \geqslant 0$*. Then the commutative square of pro-spectra*

$$\begin{array}{ccc} \text{“}\lim\text{”}\, \mathrm{E}^\mathrm{G}(\mathrm{R}_\lambda) & \longrightarrow & \text{“}\lim\text{”}\, \mathrm{E}^\mathrm{G}(\mathrm{R}_\lambda(\lambda)) \\ \downarrow & & \downarrow \\ \text{“}\lim\text{”}\, \mathrm{E}^\mathrm{G}(\mathrm{S}_\lambda) & \longrightarrow & \text{“}\lim\text{”}\, \mathrm{E}^\mathrm{G}(\mathrm{S}_\lambda(\lambda)) \end{array}$$

is cartesian.

Proof. Consider the diagram of cofiltered systems boundedly weighted categories (see Theorem 3.3.13)

$$\begin{array}{ccc} \{\mathbf{Perf}_{[\mathrm{Spec}\,\mathrm{R}_\lambda/\mathrm{G}]}\} & \longrightarrow & \{\mathbf{Perf}_{[\mathrm{Spec}\,\mathrm{R}_\lambda(\lambda)/\mathrm{G}]}\} \\ \downarrow & & \downarrow \\ \{\mathbf{Perf}_{[\mathrm{Spec}\,\mathrm{S}_\lambda/\mathrm{G}]}\} & \longrightarrow & \{\mathbf{Perf}_{[\mathrm{Spec}\,\mathrm{S}_\lambda(\lambda)/\mathrm{G}]}\}. \end{array} \tag{7.4.3}$$

All the categories in the diagram are generated by pullbacks of the set $\mathcal{P}_\mathrm{G}$ of G-equivariant finitely generated projective k-modules and all the functors are levelwise epimorphisms. Since $\mathrm{S}_\lambda(\lambda) = \mathrm{S}_\lambda \otimes_{\mathrm{R}_\lambda} \mathrm{R}_\lambda(\lambda)$ the base change map is levelwisely an isomorphism.

The map $\pi_0\mathrm{S}_\lambda \to \pi_0\mathrm{S}_\lambda(\lambda)$ is surjective, so the pullback $\mathrm{S}_\lambda \times_{\mathrm{S}_\lambda(\lambda)} \mathrm{R}_\lambda(\lambda)$ is connective. The pullback

$$\mathbf{Perf}_{[\mathrm{Spec}\,\mathrm{S}_\lambda/\mathrm{G}]} \times_{\mathbf{Perf}_{[\mathrm{Spec}\,\mathrm{S}_\lambda(\lambda)/\mathrm{G}]}} \mathbf{Perf}_{[\mathrm{Spec}\,\mathrm{R}_\lambda(\lambda)/\mathrm{G}]}$$

may be identified with $[\mathbf{Perf}_{\mathrm{S}_\lambda \times_{\mathrm{S}_\lambda(\lambda)} \mathrm{R}_\lambda(\lambda)}/\mathrm{G}]$ by [Lur18, Theorem 16.2.1.1] and so is naturally weighted in such a way that the functor

$$\mathbf{Perf}_{[\mathrm{Spec}\,\mathrm{R}_\lambda/\mathrm{G}]} \to \mathbf{Perf}_{[\mathrm{Spec}\,\mathrm{S}_\lambda/\mathrm{G}]} \times_{\mathbf{Perf}_{[\mathrm{Spec}\,\mathrm{S}_\lambda(\lambda)/\mathrm{G}]}} \mathbf{Perf}_{[\mathrm{Spec}\,\mathrm{R}_\lambda(\lambda)/\mathrm{G}]}$$

is weight exact.

It follows from our assumption and [KST17, Lemma 4.10] that the fibers of the maps

$$\{\mathrm{R}_\lambda\} \to \{\mathrm{R}_\lambda(\lambda)\} \text{ and } \{\mathrm{S}_\lambda\} \to \{\mathrm{S}_\lambda(\lambda)\}$$

are equivalent. In particular the map

$$\{\mathrm{R}_\lambda\} \to \{\mathrm{S}_\lambda \times_{\mathrm{S}_\lambda(\lambda)} \mathrm{R}_\lambda(\lambda)\}$$

is a pro-equivalence. By Lemma 7.4.1 and the map

$$\{\mathbf{Perf}_{[\mathrm{Spec}\,\mathrm{R}_\lambda/\mathrm{G}]}\} \to \{\mathbf{Perf}_{[\mathrm{Spec}\,\mathrm{S}_\lambda/\mathrm{G}]} \times_{\mathbf{Perf}_{[\mathrm{Spec}\,\mathrm{S}_\lambda(\lambda)/\mathrm{G}]}} \mathbf{Perf}_{[\mathrm{Spec}\,\mathrm{R}_\lambda(\lambda)/\mathrm{G}]}\}$$

is a pro-equivalence. So, (7.4.3) is a pro-Milnor square and the result follows from Theorem 6.4.5. □

Applying the above theorem to the map of constant families $\{\mathrm{R}\} \to \{\pi_0(\mathrm{R})\}$ we get:

Corollary 7.4.4. *Assume* R *is bounded. Under the same set-up as before with* $\mathrm{S}_\lambda = \pi_0(\mathrm{R})$ *the square*

$$\begin{array}{ccc} \mathrm{E}^{\mathrm{G}}(\mathrm{R}) & \longrightarrow & \text{“}\lim\text{”}\, \mathrm{E}^{\mathrm{G}}(\mathrm{R}_\lambda(\lambda)) \\ \downarrow & & \downarrow \\ \mathrm{E}^{\mathrm{G}}(\pi_0(\mathrm{R})) & \longrightarrow & \text{“}\lim\text{”}\, \mathrm{E}^{\mathrm{G}}(\pi_0(\mathrm{R})/s(\mathrm{M})^\lambda) \end{array}$$

is cartesian in $\mathrm{Pro}(\mathrm{Spt})$ *if* R_f *is discrete for any* $f \in s(\pi_0(\mathrm{M})) \subset \pi_0(\mathrm{R})$.

Proof. Since R is Noetherian, $\pi_i(\mathrm{R})$ are all finitely generated. Hence our assumption yields that for any $f \in s(\mathrm{M})$ there exists n such that the group $f^n\pi_i(\mathrm{R})$ is trivial. Since $s(\mathrm{M})$ is finitely generated, $s(\mathrm{M})^k\pi_i(\mathrm{R}) = 0$ for k big enough. Finally, the map “lim”$\pi_0(\mathrm{R})(\lambda) \to$ “lim”$\pi_0(\mathrm{R})/s(\mathrm{M})^\lambda$ is a pro-equivalence of pro-$\mathbb{E}_\infty$-rings by [KST17, Lemma 4.10] So by Corollary 7.4.1 we are in situation of Theorem 7.4.2. □

Remark 7.4.5. Using the results of Subsection 6.4.10 one can generalize Lemma 7.4.1, Theorem 7.4.2 and Corollary 7.4.4 to the case when the simplicial rings are not bounded (and even replace them with connective $\mathbb{E}_\infty$-ring spectra using [Lur18, Lemma 8.4.4.5]). For that one needs to use *weak* equivalences and *weak* pullback squares in the statements instead of pro-equivalences and pullbacks, and to assume that E is a connected invariant.

Now we apply Theorem 7.4.2 to the case where $\{\mathrm{R}_\lambda\}$ and $\{\mathrm{S}_\lambda\}$ are constant families of discrete Noetherian k-algebras R and S. For an equivariant ideal I of R that maps isomorphically to an equivariant ideal of S we take M to be a G-equivariant vector bundle that surjects onto I (this exists by the resolution property). By Corollary 7.4.4 applied to the discrete R we can also take the classical quotients $\mathrm{R}/\mathrm{I}^\lambda$ and $\mathrm{S}/\mathrm{I}^\lambda$ in place of the derived ones $\mathrm{R}(\lambda)$ and $\mathrm{S}(\lambda)$. Summarizing we have:

Corollary 7.4.6. *Let* G *be a linearly reductive embeddable group over a commutative ring* k. *Let* $\mathrm{R} \xrightarrow{f} \mathrm{S}$ *be an equivariant map of Noetherian commutative* k*-algebras with an action of* G, $\mathrm{I} \subset \mathrm{R}$ *be an equivariant ideal which* f *maps isomorphically onto an ideal* $\mathrm{J} \subset \mathrm{S}$. *Then the map*

$$\text{“}\lim\text{”}\, \mathrm{E}^{\mathrm{G}}(\mathrm{R}, \mathrm{I}^n) \to \text{“}\lim\text{”}\, \mathrm{E}^{\mathrm{G}}(\mathrm{S}, \mathrm{J}^n)$$

is a pro-equivalence.

7.5. **Proof of pro-cdh-descent.** We are now ready to prove Theorem 7.3.1. The proof follows step by step Section 5 of [KST17]. A reader familiar with the techniques might skip the proofs in the chapter on the first reading. A slightly different argument is only needed in the end of our proof (see 7.5.8) to replace the construction [KST17, 5.4].

Until the end of the section consider an abstract blow-up square of stacks

$$\begin{array}{ccc} \mathrm{E} & \longrightarrow & \tilde{\mathrm{X}} \\ \downarrow & & \downarrow p \\ \mathrm{Y} & \xrightarrow{i} & \mathrm{X}, \end{array}$$

that is, a pullback square with p proper representable and i closed immersion such that p induces an isomorphism $\tilde{X} - E \to X - Y$. We will assume that X is ANS. Define Y_n to be the n-th infinitesimal thickening of Y in X and E_n to be the n-th infinitesimal thickening of E in $\tilde{X}$.

Proposition 7.5.1. *Theorem 7.3.1 holds for an abstract blow-up square (7.5) with p being finite.*

Proof. Pick a Nisnevich covering $\{[U_i/G_i] \to X\}_{i \in I}$ from the definition of Nisnevich-local linearizability, where U_i is affine over an affine scheme S_i and G_i is a linearly reductive group scheme over S_i acting on U_i. By [AHR19, Corollary 13.2] we may assume that all G_i are embeddable. Hence by Theorem 7.1.6 it suffices to prove the claim for the case of X being a quotient stack [Spec(R)/G]. Here we use that taking weak pullbacks of pro-spectra as well as taking formal limits commutes with finite limits (see Lemmas 6.3.4 and Lemma 6.3.1). Since p is representable and finite and Y is a closed substack, our diagram of stacks is obtained from a diagram of G-equivariant Noetherian rings

$$\begin{array}{ccc} R & \longrightarrow & R/I \\ \downarrow{\scriptstyle \pi} & & \downarrow \\ S & \longrightarrow & S/IS \end{array}$$

by taking the quotient by G of the corresponding cdh-square of affine schemes.

By definition of cdh-square π becomes an isomorphism after localizing any $x \in I$. Hence both $\mathrm{Ker}(\pi)_x$ and $\mathrm{Coker}(\pi)_x$ are trivial. Since $\mathrm{Ker}(\pi)$ and $\mathrm{Coker}(\pi)$ are finitely generated, $x^n \mathrm{Ker}(\pi)$ and $x^n \mathrm{Coker}(\pi)$ are trivial for n big enough. Since I is finitely generated, $I^n \mathrm{Ker}(\pi) = 0$ and $I^n \mathrm{Coker}(\pi) = 0$ for n big enough. The first equality and the Artin-Rees Lemma imply that $\mathrm{Ker}(I^n \to S) = I^n \cap \mathrm{Ker}(\pi) = 0$. The second equality implies $I^n S \subset \pi(R)$. Factoring π as $R \to R/\ker(\pi) \to S$ we reduce to the cases of π being surjective and π being injective. If π is surjective then it maps I^n isomorphically onto $\pi(I^n)$. If π is injective then it maps the ideal $I^{n+1} \subset \pi^{-1}(I^{n+1}S) \subset \pi^{-1}(\pi(I)) = I$ isomorphically onto the ideal $\pi(I^n)$.

Note that replacing I in the diagram by an ideal J such that $I^k \subset J \subset I$ for some k gives rise to equivalent pro-objects, so the result follows from Corollary 7.4.6. □

Remark 7.5.2. The proof of 7.5.1 doesn't use that G_i are almost multiplicative, so finite pro-cdh-descent holds for X being any ANS stack.

Corollary 7.5.3. *Theorem 7.3.1 holds if it holds for any square (7.5) with* $X - Y \subset X$ *and* $\tilde{X} - E \subset \tilde{X}$ *being dense open.*

Proof. By Proposition 7.5.1 replacing X by the closure of $X - Y$, $\tilde{X}$ by the closure of $\tilde{X} - E$, Y and E by the corresponding pullbacks doesn't change "lim" $E(X, Y_n)$ and "lim" $E(\tilde{X}, E_n)$ up to weak equivalence. □

Proposition 7.5.4. *If Theorem 7.3.1 holds for actual blow-up squares then it holds for any abstract blow-up square.*

Proof. Let $Y' \subset Y \subset X$ be closed substacks. Denote by Y_n and Y'_n the corresponding infinitesimal thickenings of the substacks. Using the same argument as in the proof of [KST17, Claim 5.3] (and Proposition 7.5.1) we see that the theorem holds for the square

$$\begin{array}{ccc} Y \times_X \mathrm{Bl}_{Y'/X} & \longrightarrow & \mathrm{Bl}_{Y'/X} \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X. \end{array}$$

Now consider an abstract blow-up square (7.5). By Corollary 7.5.3 we may assume $\mathrm{X} - \mathrm{Y}$ to be dense in X. Using a version of flatification by blow-ups (see [Ryd17, Theorem 4.2]) we may find a sequence of blow-ups $\mathrm{X}' \to \mathrm{X}$ that factors through $\tilde{\mathrm{X}}$ and whose centers are all substacks of the preimage of Y ([HK19, Corollary 2.4]). Since $\mathrm{X}' \to \tilde{\mathrm{X}}$ is also a sequence of blow-ups the observation above implies that Theorem 7.3.1 holds for the top square and also for the composed square in the diagram

$$\begin{array}{ccc} \mathrm{Y} \times_{\mathrm{X}} \mathrm{X}' & \longrightarrow & \mathrm{X}' \\ \downarrow & & \downarrow \\ \mathrm{E} & \longrightarrow & \tilde{\mathrm{X}} \\ \downarrow & & \downarrow \\ \mathrm{Y} & \longrightarrow & \mathrm{X}. \end{array}$$

Hence it also holds for the bottom square. □

7.5.5. *Pro-cdh-descent for derived blow-ups.* Notice that although Theorem 7.3.2 yields cdh-descent for derived blow-ups, it does not automatically imply pro-cdh-descent. In this section we solve the issue.

Let X be a Noetherian quotient stack [Spec(R)/G] over k where G is embeddable and almost multiplicative linearly reductive, $\mathrm{V} \xrightarrow{s} \mathrm{R}$ be an equivariant cosection of a locally free module. Denote by Z_k the quasi-smooth closed substack of X defined by $s^{\otimes n}$. Denote by $\mathcal{X}^k$ the derived blow-up of Z_k in X and let $\mathcal{D}^k$ denote the corresponding derived exceptional divisor which is a quasi-smooth substack. We also denote by $\mathcal{D}^k_n$ its n-th nilpotent thickening. Note that the corresponding classical blow-ups over $[\mathrm{Spec}(\mathrm{Sym}_{\mathrm{R}}(\mathrm{V}))/\mathrm{G}]$ are all equivalent, so we have canonical maps $\mathcal{X}^1 \to \mathcal{X}^i$ and $\mathcal{D}^1_{in} \to \mathcal{D}^i_n$ for all $i > 0$.

These data also allow us to define the thickened semi-derived exceptional divisors $\mathcal{E}^k_n$. These are defined as the derived pullbacks of the n-th (non-derived) nilpotent thickening of the *classical* exceptional divisor of the corresponding blow-up of $[\mathrm{Spec}(\mathrm{Sym}_{\mathrm{R}}(\mathrm{V}^{\otimes k}))/\mathrm{G}]$.

Lemma 7.5.6. *Fix an positive integer k. In the notation above the commutative diagram*

$$\begin{array}{ccc} \{\mathcal{D}^1_{nk}\} & \longrightarrow & \{\mathcal{D}^k_n\} \\ \downarrow & & \downarrow \\ \mathcal{X}^1 & \longrightarrow & \mathcal{X}^k \end{array}$$

induces a weak pullback of pro-spectra after applying any connective localizing invariant.

Proof. The diagram in question is cartesian by construction. Since $\mathcal{X}^1 - \mathcal{D}^1_k \cong \mathcal{X}^k - \mathcal{D}^k \cong \mathrm{X} - \mathrm{Z}$, the associated diagram of underlying classical stacks induces a weak pullback square in pro-spectra by Proposition 7.5.1. If $\mathcal{X}_1$ were of the form $[\mathrm{Spec}\,\mathrm{R}/\mathrm{G}]$ where G is embeddable linearly reductive and R is Noetherian, then Corollary 7.4.4 would imply the result. At least the pullback of our diagram from $\mathcal{X}_1$ to any such affine quotient stack induces weak pullback square in pro-spectra. The derived stack $\mathcal{X}_k$ is a pullback of a classical blow-up from the affine quotient stack $[\mathrm{Spec}(\mathrm{Sym}_{\mathrm{R}}(\mathrm{V}^{\otimes k}))/\mathrm{G}]$. So by Theorem 7.1.12 it admits an affine Nisnevich cover by stacks of the form $[\mathrm{Spec}\,\mathrm{A}_i/\mathrm{G}_i]$ where A_i are Noetherian simplicial k-algebras and G_i are embeddable linearly reductive almost multiplicative over k acting on A_i. Now the result follows from Theorem 7.1.6. □

Lemma 7.5.7. *For any connective localizing invariant* E *the map*

$$\text{“}\lim\text{”}\, \mathrm{E}(\mathcal{D}^k_n) \to \text{“}\lim\text{”}\, \mathrm{E}(\mathcal{E}^k_n)$$

is a weak equivalence.

Proof. By definition the map $\mathcal{E}^k_n \to \mathcal{D}^k_n$ is the derived pullback of a map $\pi_0 \mathrm{D}_n \to \mathrm{D}_n$ along $\mathrm{X} \to [\mathrm{Spec}(\mathrm{Sym}_\mathrm{R}(\mathrm{V}^{\otimes k}))/\mathrm{G}]$, where D_n is the n-th nilpotent thickening of the derived exceptional divisor D on $\mathrm{Bl}_{[\mathrm{Spec}\,\mathrm{R}/\mathrm{G}]}[\mathrm{Spec}(\mathrm{Sym}_\mathrm{R}(\mathrm{V}^{\otimes k}))/\mathrm{G}]$. This admits a Nisnevich cover by stacks of the form $[\mathrm{Spec}\,\mathrm{A}_i/\mathrm{G}_i]$ by Theorem 7.1.12. So the result follows from Proposition 7.4.4 and Theorem 7.1.6. □

Corollary 7.5.8. *For any connective localizing invariant* E *the map*

$$\text{“}\lim\text{”}\,\mathrm{E}(\mathrm{X},\mathrm{Z}_n) \to \text{“}\lim\text{”}\,\mathrm{E}(\mathcal{X}^1,\mathcal{D}^1_n)$$

is a weak equivalence of pro-spectra.

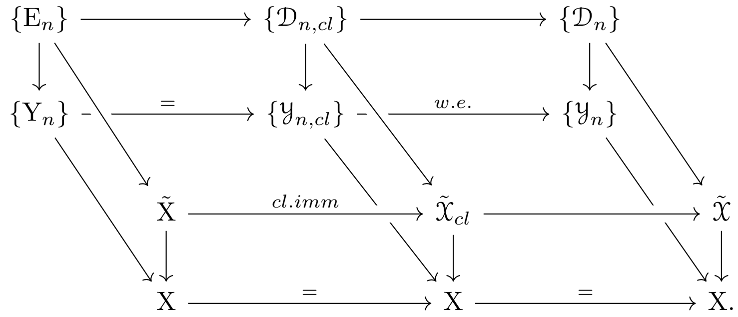
Proof. We have a commutative diagram of pro-spectra

$$
\begin{array}{ccccc}
\text{“}\lim\limits_{n}\text{”}\,\mathrm{E}(\mathrm{X},\mathrm{Z}_n) & \longrightarrow & \text{“}\lim\limits_{n}\text{”}\,\mathrm{E}(\mathcal{X}^1,\mathcal{D}^1_n) & & \\
\downarrow \cong & & \downarrow \cong & & \\
\text{“}\lim\limits_{k,nk}\text{”}\,\mathrm{E}(\mathrm{X},\mathrm{Z}_{nk}) & \longrightarrow & \text{“}\lim\limits_{k,nk}\text{”}\,\mathrm{E}(\mathcal{X}^1,\mathcal{D}^1_{nk}) & & \\
\uparrow = & & \uparrow (1) & & \\
\text{“}\lim\limits_{k,nk}\text{”}\,\mathrm{E}(\mathrm{X},\mathrm{Z}_{nk}) & \longrightarrow & \text{“}\lim\limits_{k,nk}\text{”}\,\mathrm{E}(\mathcal{X}^k,\mathcal{D}^k_n) & \xrightarrow{(2)} & \text{“}\lim\limits_{k,nk}\text{”}\,\mathrm{E}(\mathcal{X}^k,\mathcal{E}^k_n) \\
\uparrow \cong & & \uparrow \cong & & \uparrow \cong \\
\text{“}\lim\limits_{k}\text{”}\,\mathrm{E}(\mathrm{X},\mathrm{Z}_k) & \longrightarrow & \text{“}\lim\limits_{k}\text{”}\,\mathrm{E}(\mathcal{X}^k,\mathcal{D}^k) & \longrightarrow & \text{“}\lim\limits_{k}\text{”}\,\mathrm{E}(\mathcal{X}^k,\mathcal{E}^k)
\end{array}
$$

where the subscript "k, nk" signifies that the limit is taken over the poset of pairs (k, nk) with the componentwise divisibility order. The lower vertical maps are induced by the functor that sends k to (k, k). This functor is final, so the maps are equivalences. The bottom horizontal composite map is an equivalence by Theorem 7.3.2. The map (1) is an equivalence by Lemma 7.5.6 and the map (2) is an equivalence by Lemma 7.5.7, so the top horizontal map is an equivalence. □

Proof of Theorem 7.3.1. By Proposition 7.5.4 we may assume that the square (7.5) is a classical blow-up square. Pick a Nisnevich covering $\{[\mathrm{U}_i/\mathrm{G}_i] \to \mathrm{X}\}_{i\in \mathrm{I}}$ from local nice linearizability of X, where U_i is affine over an affine scheme S_i and G_i is an almost multiplicative linearly reductive group scheme over S_i acting on U_i. Moreover, we may shrink the cover further to ensure that G_i are embeddable by [AHR19, Corollary 13.2]. Finally Theorem 7.1.6 implies that it suffices to prove the claim for the case of a classical blow-up square of a quotient stack $\mathrm{X} = [\mathrm{Spec}(\mathrm{R})/\mathrm{G}]$ at a closed substack Z defined by an equivariant ideal I, where G embeddable almost multiplicative linearly reductive. Here we use that taking weak pullbacks of pro-spectra as well as taking formal limits commutes with finite limits (see Lemmas 6.3.4 and Lemma 6.3.1).

By [Gro17, Theorem A] X has the resolution property. Hence we may find a surjection $\mathrm{V} \to \mathrm{I}$, where V is a G-equivariant locally free R-module. Consider the derived stack $\mathcal{Y}$ defined as the zero locus of the cosection $\mathrm{V} \to \mathrm{I} \to \mathrm{R}$. In other words, $\mathcal{Y} = [\mathrm{Spec}(\mathrm{R} \otimes_{Sym_\mathrm{R}(\mathrm{V})} \mathrm{R})/\mathrm{G}]$. We consider the derived blow-up $\tilde{\mathcal{X}} = \mathrm{Bl}^{\mathbb{L}}_{\mathcal{Y}/\mathrm{X}}$ and its derived exceptional divisor $\mathcal{D}$. We denote the nilpotent thickenings of $\mathcal{Y}$ and $\mathcal{D}$ by $\mathcal{Y}_n$ and $\mathcal{D}_n$, respectively. These data fit into the commutative diagram



We need to prove that the leftmost face in the diagram induces a weak pullback square after taking formal limits and applying Pro(E). The top left face does induce a weak pullback square by Proposition 7.5.1 since it is a pullback square of stacks and since $\tilde{\mathrm{X}} - \mathrm{E} \cong \tilde{\mathcal{X}}_{cl} - \mathcal{D}_{cl}$. By definition $\tilde{\mathcal{X}}$ is a derived pullback of a classical blow-up from $[\mathrm{Spec}(\mathrm{S}ym_{\mathrm{R}}(\mathrm{V}))/\mathrm{G}]$, so by Theorem 7.1.12 it admits a Nisnevich cover by stacks of the form $[\mathrm{Spec}\,\mathrm{A}_i/\mathrm{G}_i]$ where A_i are Noetherian simplicial k-algebras and G_i are embeddable almost multiplicative linearly reductive over k acting on A_i. Combining the Nisnevich descent for E (see Theorem 7.1.6) and Corollary 7.4.4 we obtain that that the top right face also induces a weak pullback square after taking formal limits and applying Pro(E).

Now since the map "lim" $\mathrm{E}(\mathcal{Y}_n) \to$ "lim" $\mathrm{E}(\mathcal{Y}_{n,cl})$ is an equivalence (by Lemma 7.4.1 and [KST17, Lemma 4.10]) it suffices to show that the rightmost face in the diagram induces a weak pullback square after taking formal limits and applying Pro(E). This follows from Corollary 7.5.8. □

7.6. **Proof of the Weibel's conjecture.** We are now ready to prove Theorem 7.3.3.

Lemma 7.6.1. *Let* $\mathrm{X}' \xrightarrow{f} \mathrm{X}$ *be an infinitesimal extension of ANS stacks. Then the induced map*

$$\mathrm{K}_{-n}(\mathrm{X}) \to \mathrm{K}_{-n}(\mathrm{X}')$$

is an isomorphism for n *greater than the Nisnevich cohomological dimension of* X.

Proof. By Theorem 7.1.6 K(−) and $\mathrm{K}(- \times_{\mathrm{X}} \mathrm{X}')$ are Nisnevich sheaves of spectra on the small etale site on X. So the presheaf K^{nil} defined by the formula

$$\mathrm{U} \mapsto \mathrm{Fib}(\mathrm{K}(\mathrm{U}) \to \mathrm{K}(\mathrm{U} \times_{\mathrm{X}} \mathrm{X}'))$$

is a Nisnevich sheaf of spectra. Let $\{[\mathrm{U}_i/\mathrm{G}_i] \to \mathrm{X}\}$ be a Nisnevich covering given by local linearizability of X. By [AHR19, Corollary 13.2] we may also assume that G_i are embeddable. By Corollary 5.3.6 the spectrum $\mathrm{K}^{nil}([\mathrm{U}_i/\mathrm{G}_i])$ is connective for any i. Hence the homotopy sheaves $\pi_i^{\mathrm{N}is}(\mathrm{K}^{nil})$ are trivial for $i < 0$. The descent spectral sequence for K^{nil} has the form:

$$\mathrm{H}^p_{\mathrm{N}is}(\mathrm{X}, \pi_q^{\mathrm{N}is}\mathrm{K}^{nil}) \Rightarrow \pi_{q-p}\mathrm{K}^{nil}(\mathrm{X}).$$

The left hand side is trivial for $q < 0$ and for $p > \dim_{\mathrm{N}is}\mathrm{X}$. Hence $\pi_{-i}\mathrm{K}^{nil}(\mathrm{X}) = 0$ for $i > \dim_{\mathrm{N}is}\mathrm{X}$. □

Proof when X *has the resolution property.* We argue by induction on d. Since $d \geqslant \mathrm{cd}_{\mathrm{Nis}}(\mathrm{X})$ (Proposition 7.2.9), we may assume that X is reduced by Corollary 7.6.1. For any element $\gamma \in \mathrm{K}_{-i}(\mathrm{X})$, there exists by the killing lemma [HK19, Proposition 7.3] a sequence of blow-ups

$f : X' \to X$ with nowhere dense centers such that $f^*(\gamma) = 0$ in $K_{-i}(X')$. This fits in a proper cdh square

$$\begin{array}{ccc} Z' & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Z & \longrightarrow & X \end{array}$$

where $Z \subset X$ is any nowhere dense closed substack for which f is an isomorphism over $X \smallsetminus Z$. Let $Z(n)$ and $Z'(n)$ denote the nth infinitesimal thickenings of Z and Z', respectively. By Theorem 7.3.1,we get a long exact sequence

$$\cdots \to \text{“}\lim_n\text{”} K_{-i+1}(Z'(n)) \to K_{-i}(X) \to \text{“}\lim_n\text{”} K_{-i}(Z(n)) \oplus K_{-i}(X') \to \cdots$$

of pro-abelian groups. Now $Z(n)$ and $Z'(n)$ are noetherian ANS stacks satisfying the resolution property and of fppf-covering dimension $< d$, so “lim”$K_{-i+1}(Z'(n))$ and “lim”$K_{-i}(Z(n))$ both vanish by the induction hypothesis. It follows that $f^* : K_{-i}(X) \to K_{-i}(X')$ is injective and hence that $\gamma = 0$. $\square$

Proof of general case. We again argue by induction on d. Suppose $d = 0$. By Corollary 7.1.15 and Theorem 7.1.14, there is a convergent Nisnevich-descent spectral sequence:

$$H^p_{\mathrm{Nis}}(X, \pi_q^{\mathrm{Nis}}(K)) \Rightarrow K_{q-p}(X),$$

where $\pi_q^{\mathrm{Nis}}(K)$ denotes the Nisnevich sheaf associated with K_q. It follows from Proposition 7.2.9 and the previous case that $H^p_{\mathrm{Nis}}(X, \pi_q^{\mathrm{Nis}}(K))$ vanishes for all $q < 0$ and $p > 0$ and therefore also $K_{-i}(X) = 0$ for $i > 0$.

Now suppose $d \geqslant 1$. By Theorem 7.1.12, there is a finite sequence of open immersions $\varnothing = U_0 \hookrightarrow U_1 \hookrightarrow \cdots \hookrightarrow U_n = X$, and Nisnevich squares

$$\begin{array}{ccc} W_j & \longrightarrow & V_j \\ \downarrow & & \downarrow \\ U_{j-1} & \longrightarrow & U_j, \end{array}$$

where each W_j and V_j satisfy the resolution property and have fppf-covering dimension $\leqslant d$. We prove by induction on j that for $i > d$, $K_{-i}(U_j)$ vanishes. For $j = 0$, this follows from the previous case. By Proposition 7.2.9 and Corollary 7.6.1, we may assume that U_j is reduced. Choose $\gamma \in K_{-i}(U_j)$. By induction on j, and the previous case for stacks with resolution property, the groups $K_{-l}(W_j)$, $K_{-l}(V_j)$ and $K_{-l}(U_{j-1})$ vanish for all $l > d$. By Nisnevich descent (Corollary 7.1.15), we have a long exact sequence:

$$\cdots \to K_{-i+1}(W_j) \xrightarrow{\partial} K_{-i}(U_j) \to K_{-i}(U_{j-1}) \oplus K_{-i}(V_j) \to \cdots .$$

By induction hypothesis on j, we deduce that $\gamma = \partial(\alpha)$ for some $\alpha \in K_{-i+1}(W_j)$. By applying the killing lemma [HK19, Proposition 7.3] to the étale morphism $W_j \to U_j$, we can find a sequence of blow-ups $f : U'_j \to U_j$ with nowhere dense centers such that for the induced map $f_W : W'_J := U'_j \times_{U_j} W_j \to W_j$, $f_W^*(\alpha) = 0$ in $K_{-i+1}(W'_j)$. Since U'_j is again a noetherian ANS stack (Lemma 7.1.11), we conclude that $f^*(\gamma) = f^*(\partial(\alpha)) = \partial(f_W^*(\alpha)) = 0$. Now as in the first case, by using Theorem 7.3.1 and the induction hypothesis on d, we conclude that $\gamma = 0$. $\square$

7.7. **Applications to truncating invariants of stacks.** We will say that a closed immersion of derived algebraic stacks $X \to Y$ is a **nilpotent extension** if the ideal sheaf is nilpotent. The next result is a global version of Corollary 1.4.7.

Theorem 7.7.1. *Let* Y *be an ANS derived algebraic stack and let* $X \to Y$ *be a nilpotent extension of derived algebraic stacks. Then the induced map* $E(Y) \to E(X)$ *is an equivalence for any truncating invariant* E.

Proof. Let us first prove the result assuming that Y is a global quotient stack [Spec R/G] where G is a an embeddable, nice group scheme over an affine scheme B and Spec R is a derived B-scheme. In this case, we can form the cartesian square

$$\begin{array}{ccc} \mathrm{X}' & \longrightarrow & \operatorname{Spec} \mathrm{R} \\ \big\downarrow & & \big\downarrow{\scriptstyle p} \\ \mathrm{X} & \longrightarrow & \mathrm{Y}, \end{array}$$

where $\operatorname{Spec} \mathrm{R} \to \mathrm{Y}$ is the quotient map. Now, a closed immersion is an affine morphism and thus $\mathrm{X}' \to \operatorname{Spec} \mathrm{R}$ is an affine morphism. Therefore, we deduce that $\mathrm{X}' \cong \operatorname{Spec} \mathrm{S}$. On the other hand, $\operatorname{Spec} \mathrm{R} \to \mathrm{Y}$ is G-torsor and thus $\mathrm{X}' \to \mathrm{X}$ is a G-torsor as well and thus we conclude that $\mathrm{X} = [\operatorname{Spec} \mathrm{S}/\mathrm{G}]$. The result, in this case, then follows from Corollary 1.4.7.

To prove the result in general, we consider a decomposition of Y as in Theorem 7.1.12) and induct on n which exists by point (2) above. More precisely, we say that an ANS derived algebraic stack X is of *length at most* n if it has a decomposition of the form (7.1.13) of length n. With this terminology, a global quotient stack is length at most zero and we have proved the result in this case in the previous paragraph (see Proposition 5.4.11). Let us assume that the result has been proved for all ANS derived algebraic stacks of length at most $n-1$ and let Y be an ANS derived algebraic stack of length at most n. Choose a decomposition of the form (7.1.13) as above for Y. We have a morphism of cartesian squares induced by pullback along $\mathrm{X} \to \mathrm{Y}$.

$$\begin{array}{ccc} \mathrm{W}_n & \longrightarrow & \mathrm{V}_n \\ \big\downarrow & & \big\downarrow \\ \mathrm{U}_{n-1} & \longrightarrow & \mathrm{Y} \end{array} \quad \Leftarrow \quad \begin{array}{ccc} \mathrm{W}'_n & \longrightarrow & \mathrm{V}'_n \\ \big\downarrow & & \big\downarrow \\ \mathrm{U}'_{n-1} & \longrightarrow & \mathrm{X} \end{array}$$

Since all the morphisms above are representable and have affine diagonal, Theorem 7.1.12 tells us that all the derived stacks are ANS. By construction, U_{n-1} is length at most $n-1$. Therefore, by induction hypothesis, E converts the map $\mathrm{U}'_{n-1} \to \mathrm{U}_{n-1}$ to an equivalence. Now E also converts the map $\mathrm{V}'_n \to \mathrm{V}_n$ to an equivalence since V_n is a global quotient stack. Lastly, W_n is again an ANS derived algebraic stack of length at most $n-1$ (we can pullback a decomposition of U_{n-1} to one on W_n) and thus the result also applies by induction hypothesis. Since E converts Nisnevich squares of ANS derived stacks to bicartesian square of spectra, the result is proved.

□

Let X be a derived algebraic stack, then we have its classical locus $\mathrm{X}^{\mathrm{cl}} \hookrightarrow \mathrm{X}$; this morphism is a closed immersion and X^{cl} is a classical algebraic stack. A global version of Example (5.4.13) is as follows

Corollary 7.7.2. *Let* X *be an ANS derived algebraic stack. Then for any truncating invariant* E*, we have an induced equivalence*

$$\mathrm{E}(\mathrm{X}^{\mathrm{cl}}) \simeq \mathrm{E}(\mathrm{X}).$$

7.7.3. *Cdh excision for truncating invariants.* In the next situation, we content ourselves with *classical* algebraic stacks. As before let

$$\begin{array}{ccc} \mathrm{Z} & \longrightarrow & \tilde{\mathrm{X}} \\ \big\downarrow & & \big\downarrow{\scriptstyle p} \\ \mathrm{Y} & \xrightarrow{\ i\ } & \mathrm{X}, \end{array}$$

be an **abstract blow-up square** of *Noetherian* algebraic stacks. Define Y_n to be the n-th infinitesimal thickening of Y in X and Z_n to be the n-th infinitesimal thickening of Z in $\tilde{\mathrm{X}}$.

Applying Theorem 7.7.1 to the maps $\mathrm{Z} \to \mathrm{Z}_n$ and $\mathrm{Y} \to \mathrm{Y}_n$ we see that the pro-systems in Theorem 7.3.1 are actually constant. In particular we obtain the following cdh-descent result for ANS algebraic stacks:

Corollary 7.7.4. *In the notation of Theorem 7.3.1, the square*

$$\begin{array}{ccc} \mathrm{E(X)} & \longrightarrow & \mathrm{E(Y)} \\ \downarrow & & \downarrow \\ \mathrm{E}(\tilde{\mathrm{X}}) & \longrightarrow & \mathrm{E(Z)} \end{array}$$

is a pullback of spectra whenever E *is a truncating invariant. Therefore truncating invariants of ANS algebraic stacks satisfy cdh-descent.*

Proof. The only point that needs explanation is the last part. We note that cdh-descent on stacks is equivalent to a combination of Nisnevich excision and excision for abstract blowup squares; see [HK19] for details. □

Example 7.7.5. According to [LT19, Proposition 3.14], the homotopy K-theory functor is a truncating invariant of $\mathbb{Z}$-linear stable ∞-categories. Setting $\mathrm{E} = \mathrm{KH}$ in Corollary 7.7.4 reproves the main result of Hoyois and Krishna [HK19]. Of course, Corollary 7.7.4 also proves cdh-descent on stacks for other truncating invariants discussed in [LT19] such as $\mathrm{K}^{\mathrm{inf}}_{\mathbb{Q}}$ [LT19, Corollary 3.9] and periodic cyclic homology HP [LT19, Corollary 3.11] on stacks over characteristic zero rings. We note that we can apply our results in the linear setting by Remark 5.4.5.

7.8. **Applications to the lattice conjecture.** The next application is a contribution on the literature surrounding the lattice conjecture for topological K-theory in the sense of Blanc. We recall some terminology and refer the reader to [Bla16, Kon21] for details. We work in the context of $\mathbf{Cat}^{\mathrm{perf}}_{\mathbb{C}} := \mathbf{Mod}_{\mathbf{Perf}_{\mathbb{C}}}(\mathbf{Cat}^{\mathrm{perf}}_{\infty})$, the ∞-category of small stable idempotent complete $\mathbb{C}$-linear stable ∞-categories and an object of $\mathbf{Cat}^{\mathrm{perf}}_{\mathbb{C}}$ will hereon be referred to as a **$\mathbb{C}$-linear category**. We refer to [LT19, Remark 1.18] for a discussion of localizing invariants in this context and [HSS17] for details. In any case, we use the following terminology from [LT19, Remark 1.18]: a functor $\mathrm{E} : \mathbf{Cat}^{\mathrm{perf}}_{\mathbb{C}} \to \mathrm{Spt}$ which takes exact sequences in $\mathbf{Cat}^{\mathrm{perf}}_{\mathbb{C}}$ to cofiber sequences is a **$\mathbb{C}$-linear localizing invariant**. It is furthermore **truncating** if for all connective $\mathbb{E}_1$-$\mathbb{C}$-algebras (equivalently connective $\mathbb{C}$-dga's [Lur18, Proposition 25.1.2.2]) A, the map $\mathrm{E(A)} \to \mathrm{E}(\pi_0(\mathrm{A}))$ is an equivalence.

The work of Blanc [Bla16] constructs the functor of **topological K-theory**

$$\mathrm{K}^{\mathrm{top}} : \mathbf{Cat}^{\mathrm{perf}}_{\mathbb{C}} \to \mathrm{Spt}.$$

This functor is a localizing invariant [Bla16, Theorem 1.1(c)]. If X is a separated $\mathbb{C}$-scheme of finite type then we have a canonical equivalence of spectra [Bla16, Theorem 1.1(b)]

$$\mathrm{K}^{\mathrm{top}}(\mathbf{Perf}_{\mathrm{X}}) \simeq \mathrm{KU}(\mathrm{X}(\mathbb{C})),$$

where the right-hand-side is the complex topological K-theory spectrum of the analytic space associated to the $\mathbb{C}$-points of X. The lattice conjecture concerns a map of localizing invariants from $\mathrm{K}^{\mathrm{top}}$ to periodic cyclic homology called the **Chern character** [Bla16, Theorem 1.1(d)]

$$\mathrm{Ch} : \mathrm{K}^{\mathrm{top}} \to \mathrm{HP}.$$

The conjecture appears as [Bla16, Conjecture 1.7].

Conjecture 7.8.1. *Let $\mathcal{C}$ be a smooth and proper $\mathbb{C}$-linear category. Then the map*

$$\mathrm{Ch} \otimes \mathbb{C} : \mathrm{K}^{\mathrm{top}}(\mathcal{C}) \otimes \mathbb{C} \to \mathrm{HP}(\mathcal{C})$$

is an equivalence.

Conjecture 7.8.1 belongs to the world of noncommutative Hodge theory and implies the existence of a suitable Hodge structure on the homotopy groups of $\mathrm{K}^{\mathrm{top}}$; see, for example, [Per20b, Proposition 5.4, Remark 5.5]. Recent work of Konovalov brought to bear trace methods into this problem [Kon21]. His main insight is the following result.

Theorem 7.8.2 (Konovalov)**.** *Let $\mathcal{L}$ be the fiber of the (complexified) Chern character*

$$\mathcal{L}(-) := \mathrm{Fib}(\mathrm{Ch}\otimes\mathbb{C} : \mathrm{K}^{\mathrm{top}}(-)\otimes\mathbb{C} \to \mathrm{HP}(-)).$$

Then $\mathcal{L}$ is a $\mathbb{C}$-linear truncating invariant.

Now combining our results with the results of Halpern-Leistner and Pomerleano [HLP20], which verify the lattice conjecture for some classical algebraic stacks, we get:

Theorem 7.8.3. *Let X be a derived stack over $\mathbb{C}$ satisfying any of the following hypotheses:*

(1) it is of the form $[\mathrm{Y}/\mathrm{G}]$ where Y is a derived, affine G-scheme over $\mathbb{C}$, G is a reductive $\mathbb{C}$-group scheme such that Y^{cl} is a smooth $\mathbb{C}$-scheme;

(2) it is ANS derived stack such that X^{cl} is a smooth $\mathbb{C}$-stack.

Then, the lattice conjecture holds for $\mathbf{Perf}_{\mathrm{X}}$.

Proof. Applying Corollary 1.4.7 and Example 3.3.10 (for case (1)), and Theorem 7.7.1 (for (2)), in conjunction with Theorem 7.8.2, we reduce to showing that $\mathcal{L}(\mathrm{X}) = 0$ for $\mathrm{X} = \mathrm{X}^{\mathrm{cl}}$. Note that we can apply our results in the $\mathbb{C}$-linear setting by Remark 5.4.5.

In case (1) is satisfied, the result follows from [HLP20, Theorem 3.20] whose hypotheses are verified under the hypotheses on Y^{cl} by [HLP20, Theorem 2.3].

In case (2) is satisfied, an inductive argument on length of the ANS stack X, as in the proof of Theorem 7.7.1, together with Nisnevich descent for $\mathcal{L}$ reduces to the case (1). □

Conclusion

We have proved some structural results about weighted stable ∞-categories and localizing invariants applied to those. We demonstrated the usefulness of the approach by translating these results into the equivariant version of the DGM theorem, pro-excision, and pro-cdh-excision results for stacks. We also proved a version of the Weibel's conjecture on the vanishing of the negative K-groups and verified the lattice conjecture of Blanc for a large class of derived algebraic stacks using a recent result of Konovalov [Kon21].

We think that these results are interesting on their own and also provide evidence for the usefulness of weights in the context of noncommutative geometry. We are hoping to extend these methods further to use them for understanding other phenomena in the derived algebraic geometry and the geometry of stacks. We list some of the results that we expect to obtain using these methods in future works:

(1) A version of the Gabber's rigidity theorem for stacks with *nice* stabilizers extending [NR20];

(2) Pro-excision and pro-cdh-excision for hermitian K-theory using the results of [CDH+20a], [CDH+20b] and [CDH+20c].

Our methods have serious limitations, as a weight structure only exists in a specific situation. However, many interesting examples arise as certain limits of boundedly weighted stable ∞-categories. We would like to have a systematic way of dealing with such examples. We finish this thesis with the following questions that we hope to answer:

- Does there exist a combinatorially-defined notion of a semi-weighted stable ∞-category that captures those stable ∞-categories that are nice limits of weighted stable ∞-categories?
- What structure on stable ∞-category captures that it is a derived category of an exact ∞-category? Can the results we proved about additive ∞-categories be generalized to results about exact ∞-categories?

References

[AGH19] Benjamin Antieau, David Gepner, and Jeremiah Heller. K-theoretic obstructions to bounded t-structures. *Inventiones mathematicae*, 216(1):241–300, Jan 2019.

[AHHLR] Jarod Alper, Jack Hall, Daniel Halpern-Leistner, and David Rydh. Artin algebraization for pairs and applications to the local structure of stacks and ferrand pushout. *In preparation.*

[AHR19] Jarod Alper, Jack Hall, and David Rydh. The étale local structure of algebraic stacks. 2019.

[AHW17] Aravind Asok, Marc Hoyois, and Matthias Wendt. Affine representability results in a^1 -homotopy theory, i: Vector bundles. *Duke Mathematical Journal*, 166(10):1923–1953, Jul 2017.

[Alp13] Jarod Alper. Good moduli spaces for Artin stacks. *Ann. Inst. Fourier (Grenoble)*, 63(6):2349–2402, 2013.

[Alp14] Jarod Alper. Adequate moduli spaces and geometrically reductive group schemes. *Algebr. Geom.*, 1(4):489–531, 2014.

[Aok19] Ko Aoki. The weight complex functor is symmetric monoidal. *arXiv preprint arXiv:1904.01384*, 2019.

[AOV08] Dan Abramovich, Martin Olsson, and Angelo Vistoli. Tame stacks in positive characteristic. *Ann. I. Fourier*, 58(4):1057–1091, 2008.

[Bar16] Clark Barwick. On the algebraic K-theory of higher categories. *J. Topol.*, 9(1):245–347, 2016.

[BBD82] Alexander Beilinson, Joseph Bernstein, and Pierre Deligne. Faisceaux pervers. *Asterisque*, 100:5–171, 1982.

[BCKW19] Ulrich Bunke, Denis-Charles Cisinski, Daniel Kasprowski, and Christoph Winges. Controlled objects in ∞-categories and the novikov conjecture. 11 2019.

[BCM20] Bhargav Bhatt, Dustin Clausen, and Akhil Mathew. Remarks on K(1)-local K-theory. *Selecta Math. (N.S.)*, 26(3):Paper No. 39, 16, 2020.

[BGT13] Andrew J Blumberg, David Gepner, and Gonçalo Tabuada. A universal characterization of higher algebraic K-theory. *Geom. Topol.*, 17(2):733–838, 2013.

[BH18] Tom Bachmann and Marc Hoyois. Norms in motivic homotopy theory. To appear in Asterisqué, 2018.

[Bha14] Bhargav Bhatt. Algebraization and tannaka duality. *Cambridge Journal of Mathematics*, 4, 04 2014.

[BHH17] Ilan Barnea, Yonatan Harpaz, and Geoffroy Horel. Pro-categories in homotopy theory. *Algebr. Geom. Topol.*, 17(1):567–643, 2017.

[BI15] M. V. Bondarko and M. A. Ivanov. On Chow weight structures for *cdh*-motives with integral coefficients. *Algebra i Analiz*, 27(6):14–40, 2015.

[BL14] Clark Barwick and Tyler Lawson. Regularity of structured ring spectra and localization in K-theory. *arXiv preprint arXiv:1402.6038*, 2014.

[Bla16] Anthony Blanc. Topological K-theory of complex noncommutative spaces. *Compos. Math.*, 152(3):489–555, 2016.

[Bon10] M. V. Bondarko. Weight structures vs. t-structures; weight filtrations, spectral sequences, and complexes (for motives and in general). *J. K-Theory*, 6(3):387–504, 2010.

[Bon11] M. V. Bondarko. $\mathbb{Z}[1/p]$-motivic resolution of singularities. *Compos. Math.*, 147(5):1434–1446, 2011.

[Bon14] M. V. Bondarko. Weights for relative motives: Relation with mixed complexes of sheaves. *International Mathematics Research Notices*, 2014(17):4715–4767, 2014.

[Bon18] Mikhail Bondarko. On torsion pairs, (well generated) weight structures, adjacent t-structures, and related (co)homological functors. *arXiv preprint arXiv:1611.00754v6*, 2018.

[BP21] Federico Binda and Mauro Porta. Gaga problems for the brauer group via derived geometry. 2021.

[BS18a] Mikhail Bondarko and Vladimir Sosnilo. On constructing weight structures and extending them to idempotent completions. *Homology, Homotopy and Applications*, 20:37–57, 01 2018.

[BS18b] Mikhail V. Bondarko and Vladimir A. Sosnilo. Non-commutative localizations of additive categories and weight structures. *J. Inst. Math. Jussieu*, 17(4):785–821, 2018.

[CDH+20a] Baptiste Calmès, Emanuele Dotto, Yonatan Harpaz, Fabian Hebestreit, Markus Land, Kristian Moi, Denis Nardin, Thomas Nikolaus, and Wolfgang Steimle. Hermitian K-theory for stable ∞-categories i: Foundations, 2020.

[CDH$^+$20b] Baptiste Calmès, Emanuele Dotto, Yonatan Harpaz, Fabian Hebestreit, Markus Land, Kristian Moi, Denis Nardin, Thomas Nikolaus, and Wolfgang Steimle. Hermitian K-theory for stable ∞-categories ii: Cobordism categories and additivity, 2020.

[CDH$^+$20c] Baptiste Calmès, Emanuele Dotto, Yonatan Harpaz, Fabian Hebestreit, Markus Land, Kristian Moi, Denis Nardin, Thomas Nikolaus, and Wolfgang Steimle. Hermitian K-theory for stable ∞-categories iii: Grothendieck-witt groups of rings, 2020.

[Cis19] Denis-Charles Cisinski. *Higher categories and homotopical algebra*, volume 180 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2019.

[CM19] Dustin Clausen and Akhil Mathew. Hyperdescent and étale K-theory. 2019.

[DI87] Pierre Deligne and Luc Illusie. Relèvements modulo p 2 et décomposition du complexe de de rham. *Inventiones mathematicae*, 89(2):247–270, 1987.

[Dot18] Emanuele Dotto. A Dundas-Goodwillie-McCarthy theorem for split square-zero extensions of exact categories. In *An alpine bouquet of algebraic topology*, volume 708 of *Contemp. Math.*, pages 123–138. Amer. Math. Soc., Providence, RI, 2018.

[Dri02] V. Drinfeld. Dg quotients of dg categories. *Journal of Algebra*, 272:643–691, 2002.

[EK20] Elden Elmanto and Håkon Kolderup. On modules over motivic ring spectra. *Ann. K-Theory*, 5(2):327–355, 2020.

[ES21] Elden Elmanto and Vladimir Sosnilo. On nilpotent extensions of ∞-categories and the cyclotomic trace. *International Mathematics Research Notices*, 07 2021. rnab179.

[Fal88] Gerd Faltings. p-adic hodge theory. *Journal of the American Mathematical Society*, 1(1):255–299, 1988.

[Fon18] Ernest E. Fontes. Weight structures and the algebraic K-theory of stable ∞-categories. 2018.

[FV00] Eric M. Friedlander and Vladimir Voevodsky. Bivariant cycle cohomology. In *Cycles, transfers and motivic homology*. Princeton University Press, 2000.

[GGN15] David Gepner, Moritz Groth, and Thomas Nikolaus. Universality of multiplicative infinite loop space machines. *Algebr. Geom. Topol.*, 15(6):3107–3153, 2015.

[GR14] Dennis Gaitsgory and Nick Rozenblyum. Dg indschemes. *Contemporary Mathematics*, page 139–251, 2014.

[Gro17] Philipp Gross. Tensor generators on schemes and stacks. *Algebraic Geometry*, page 501–522, Sep 2017.

[Hau15] Rune Haugseng. Rectification of enriched ∞–categories. *Algebraic & Geometric Topology*, 15(4):1931–1982, Sep 2015.

[Hel20] Aron Heleodoro. Determinant map for the prestack of Tate objects. *Selecta Math. (N.S.)*, 26(5):Paper No. 76, 57, 2020.

[HK19] Marc Hoyois and Amalendu Krishna. Vanishing theorems for the negative K-theory of stacks. *Annals of K-Theory*, 4(3):439–472, Dec 2019.

[HLP14] Daniel Halpern-Leistner and Anatoly Preygel. Mapping stacks and categorical notions of properness. 2014.

[HLP20] Daniel Halpern-Leistner and Daniel Pomerleano. Equivariant Hodge theory and noncommutative geometry. *Geom. Topol.*, 24(5):2361–2433, 2020.

[Hoy18] Marc Hoyois. K-theory of dualizable categories. available at `http://www.mathematik.ur.de/hoyois/papers/efimov.pdf`, 2018.

[HR15a] Jack Hall and David Rydh. Algebraic groups and compact generation of their derived categories of representations. *Indiana Univ. Math. J.*, 64(6):1903–1923, 2015.

[HR15b] Jack Hall and David Rydh. Perfect complexes on algebraic stacks. 2015.

[HSS17] Marc Hoyois, Sarah Scherotzke, and Nicolò Sibilla. Higher traces, noncommutative motives, and the categorified Chern character. *Adv. Math.*, 309:97–154, 2017.

[Isa] A model structure on the category of pro-simplicial sets. 353.

[Iwa18] Isamu Iwanari. Tannaka duality and stable infinity-categories. *Journal of Topology*, 11(2):469–526, 2018.

[Jø10] Peter Jørgensen. Amplitude inequalities for differential graded modules. *Forum Mathematicum*, 22(5), Jan 2010.

[Kal08] Dmitry Kaledin. Non-commutative hodge-to-de rham degeneration via the method of deligne-illusie. *Pure and Applied Mathematics Quarterly*, 4(3):785–876, 2008.

[Kal17] D. Kaledin. Spectral sequences for cyclic homology. *Progress in Mathematics*, page 99–129, 2017.

[Kel17] Shane Kelly. Voevodsky motives and $l$dh descent. *Astérisque*, 391, 2017.

[Kha18] Adeel A. Khan. Algebraic K-theory of quasi-smooth blow-ups and cdh descent, 2018. to appear in Annales Henri Lebesgue.

[KKM19] D. B. Kaledin, A. A. Konovalov, and K. O. Magidson. Spectral algebras and non-commutative hodge-to-de rham degeneration. *Proceedings of the Steklov Institute of Mathematics*, 307(1):51–64, Nov 2019.

[KKP08] L. Katzarkov, M. Kontsevich, and T. Pantev. Hodge theoretic aspects of mirror symmetry. 78:87–174, 2008.

[KO10] Amalendu Krishna and Paul-Arne Ostvaer. Nisnevich descent for deligne-mumford stacks, 2010.
[Kon21] Andrey Konovalov. Nilpotent invariance of semi-topological K-theory of dg-algebras and the lattice conjecture. 2021.
[KR19] Adeel A. Khan and David Rydh. Virtual cartier divisors and blow-ups. 2019.
[KS16] Moritz Kerz and Florian Strunk. On the vanishing of negative homotopy K-theory. 2016.
[KST17] Moritz Kerz, Florian Strunk, and Georg Tamme. Algebraic K-theory and descent for blow-ups. *Inventiones mathematicae*, 211(2):523–577, Aug 2017.
[KST18] Moritz Kerz, Shuji Saito, and Georg Tamme. K-theory of non-archimedean rings i. *arXiv*, 2018.
[LMB18] Gérard Laumon and Laurent Moret-Bailly. *Champs algébriques*, volume 39. Springer, 2018.
[LMT20] Markus Land, Lennart Meier, and Georg Tamme. Vanishing results for chromatic localizations of algebraic K-theory. 2020.
[LT19] Land and Tamme. On the K-theory of pullbacks. *Annals of Mathematics*, 190(3):877, 2019.
[Lun10] Valery A. Lunts. Categorical resolution of singularities. *Journal of Algebra*, 323(10):2977–3003, May 2010.
[Lur17a] Jacob Lurie. Higher algebra. September 2017.
[Lur17b] Jacob Lurie. Higher topos theory. April 2017.
[Lur18] Jacob Lurie. Spectral algebraic geometry. February 2018.
[Mat20] Akhil Mathew. Kaledin's degeneration theorem and topological hochschild homology. *Geometry & Topology*, 24(6):2675–2708, Dec 2020.
[Mil13] James S. Milne. Lectures on étale cohomology. 2013.
[ML13] Saunders Mac Lane. *Categories for the working mathematician*, volume 5. Springer Science & Business Media, 2013.
[Mor18] Matthew Morrow. Pro unitality and pro excision in algebraic K-theory and cyclic homology. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2018(736):95–139, Mar 2018.
[MVW06] Carlo Mazza, Vladimir Voevodsky, and Charles Weibel. *Lecture Notes on Motivic Cohomology*, volume 2 of *Clay Mathematics Monographs*. AMS, 2006.
[Nee92] Amnon Neeman. The connection between the K-theory localization theorem of Thomason, Trobaugh and Yao and the smashing subcategories of Bousfield and Ravenel. *Ann. Sci. École Norm. Sup. (4)*, 25(5):547–566, 1992.
[NR20] Niko Naumann and Charanya Ravi. Rigidity in equivariant algebraic K-theory. *Annals of K-Theory*, 5(1):141–158, Mar 2020.
[NS18] Thomas Nikolaus and Peter Scholze. On topological cyclic homology. *Acta Math.*, 221(2):203–409, 2018.
[Orl97] Dmitri Orlov. Equivalences of derived categories and K3 surfaces. *J. Math. Sci. (New York)*, 84:1361–1381, 1997.
[Orl16] Dmitri Orlov. Smooth and proper noncommutative schemes and gluing of dg categories. *Advances in Mathematics*, 302:59–105, Oct 2016.
[Per20a] Alexander Perry. The integral hodge conjecture for two-dimensional calabi-yau categories. 2020.
[Per20b] Alexander Perry. The integral hodge conjecture for two-dimensional calabi-yau categories. 2020.
[Rio05] Joël Riou. Dualité de Spanier–Whitehead en géométrie algébrique. *C. R. Math. Acad. Sci. Paris*, 340(6):431–436, 2005.
[Rob80] Paul Roberts. *Homological invariants of modules over commutative rings*, volume 72 of *Séminaire de Mathématiques Supérieures [Seminar on Higher Mathematics]*. Presses de l'Université de Montréal, Montreal, Que., 1980.
[Rob13] Marco Robalo. Noncommutative motives ii: K-theory and noncommutative motives, 2013.
[Ryd17] David Rydh. Equivariant flatification, etalification and compactification. *In preparation*, 2017.
[Seg74] Graeme Segal. Categories and cohomology theories. *Topology*, 13(3):293–312, 1974.
[Ser68] Jean-Pierre Serre. *Corps locaux*. Publications de l'institut mathématique de l'université de Nancago, 1968.
[Sos19] Vladimir Sosnilo. Theorem of the heart in negative K-theory for weight structures. *Doc. Math.*, 24:2137–2158, 2019.
[Sos21] Vladimir Sosnilo. Regularity of spectral stacks and discreteness of weight-hearts. *The Quarterly Journal of Mathematics*, 03 2021. haab017.
[SS03] Stefan Schwede and Brooke Shipley. Stable model categories are categories of modules. *Topology*, 42:103–153, 01 2003.
[Sus17] Andrei Suslin. Motivic complexes over nonperfect fields. *Annals of K-Theory*, 2:277–302, 01 2017.
[Tab09] Gonçalo Tabuada. Non-commutative André-Quillen cohomology for differential graded categories. *J. Algebra*, 321(10):2926–2942, 2009.
[Tab19] Gonçalo Tabuada. A note on grothendieck's standard conjecture of type c^+ and d in positive characteristic. *Proceedings of the American Mathematical Society*, 146:5039–5054, 09 2019.
[Tam18] Georg Tamme. Excision in algebraic K-theory revisited. *Compos. Math.*, 154(9):1801–1814, 2018.
[The17] The Stacks Project Authors. The stacks project. 2017.
[Tho97] R.W. Thomason. The classification of triangulated subcategories. *Compositio Mathematica*, 1997.

[Toë07] Bertrand Toën. The homotopy theory of dg-categories and derived morita theory. *Invent. math.*, 167:615–667, 2007.

[TT90] R. W. Thomason and T. Trobaugh. Higher algebraic K-theory of schemes and of derived categories. In *The Grothendieck Festschrift III*, volume 88 of *Progress in Mathematics*, pages 247–435. Birkhäuser, 1990.

[Voe00] Vladimir Voevodsky. Triangulated categories of motives over a field. In *Cycles, transfers, and motivic homology theories*, volume 143 of *Ann. of Math. Stud.*, pages 188–238. Princeton Univ. Press, Princeton, NJ, 2000.

[Voe02] Vladimir Voevodsky. Motivic cohomology groups are isomorphic to higher chow groups in any characteristic. *Int. Math. Res. Notices*, (7):351–355, 2002.

[Voe10a] Vladimir Voevodsky. Homotopy theory of simplicial sheaves in completely decomposable topologies. *J. Pure Appl. Algebra*, 214(8):1384–1398, 2010.

[Voe10b] Vladimir Voevodsky. Unstable motivic homotopy categories in Nisnevich and cdh-topologies. *J. Pure Appl. Algebra*, 214(8):1399–1406, 2010.

[Wei13] C. Weibel. The K-book: An introduction to algebraic k-theory. 2013.